**УДК 512.554.35**

*A. Petravchuk, K. Sysak*

DOI: https://doi.org/10.18523/2617-7080220196-10

# SOLVABLE LIE ALGEBRAS OF DERIVATIONS OF RANK ONE

*Let $\mathbb{K}$ be a field of characteristic zero, $A = \mathbb{K}[x_1, \dots, x_n]$ the polynomial ring and $R = \mathbb{K}(x_1, \dots, x_n)$ the field of rational functions in $n$ variables over $\mathbb{K}$. The Lie algebra $W_n(\mathbb{K})$ of all $\mathbb{K}$-derivations on $A$ is of great interest since its elements may be considered as vector fields on $\mathbb{K}^n$ with polynomial coefficients. If $L$ is a subalgebra of $W_n(\mathbb{K})$, then one can define the rank $\mathrm{rk}_A L$ of $L$ over $A$ as the dimension of the vector space $RL$ over the field $R$. Finite dimensional (over $\mathbb{K}$) subalgebras of $W_n(\mathbb{K})$ of rank 1 over $A$ were studied by the first author jointly with I. Arzhantsev and E. Makedonskiy. We study solvable subalgebras $L$ of $W_n(\mathbb{K})$ with $\mathrm{rk}_A L = 1$, without restrictions on dimension over $\mathbb{K}$. Such Lie algebras are described in terms of Darboux polynomials.*

**Keywords:** Lie algebra, solvable Lie algebra, derivation, Darboux polynomial, polynomial ring.

## Introduction

Let $\mathbb{K}$ be a field of characteristic zero and $A = \mathbb{K}[x_1, \dots, x_n]$ the polynomial ring over $\mathbb{K}$. A $\mathbb{K}$-derivation $D$ on $A$ is a $\mathbb{K}$-linear operator $D\colon A \to A$ satisfying the Leibniz's rule: $D(fg) = D(f)g + fD(g)$ for all $f,\ g \in A$. If $D_1,\ D_2$ are $\mathbb{K}$-derivations on $A$ and $h \in A$, then $D_1 + D_2$, $hD_1$ and $[D_1, D_2] = D_1D_2 - D_2D_1$ are also derivations on $A$. The set $W_n(\mathbb{K})$ of all $\mathbb{K}$-derivations on the polynomial ring $A$ is a Lie algebra over $\mathbb{K}$ (with respect to the Lie bracket $[D_1, D_2]$) and simultaneously a free module over the polynomial ring $A$. The set of partial derivations

$$\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\}$$

forms the standard basis of the $A$-module $W_n(\mathbb{K})$. The Lie algebra $W_n(\mathbb{K})$ is of great interest since its elements may be considered as vector fields on $\mathbb{K}^n$ with polynomial coefficients (see, for example, [2–5]).

Let $R = \mathbb{K}(x_1, \dots, x_n)$ be the field of rational functions in $n$ variables over $\mathbb{K}$. Each derivation $D \in W_n(\mathbb{K})$ can be uniquely extended to a derivation on $R$ by the rule:

$$D\left(\frac{a}{b}\right) = \frac{D(a)b - aD(b)}{b^2}.$$

If $L$ is a subalgebra of the Lie algebra $W_n(\mathbb{K})$, then one can define the rank $\mathrm{rk}_A L$ of $L$ over $A$ as the dimension $\dim_R RL$ of the vector space $RL$ consisting of all linear combinations of elements $aD$, where $a \in R, D \in L$. Note that $RL$ is a Lie algebra over $\mathbb{K}$ but not a Lie algebra over $R$ in general.

Finite dimensional subalgebras of rank 1 over $A$ of the Lie algebra $W_n(\mathbb{K})$ were described in [1]. We study solvable subalgebras (without any restrictions on dimension) of rank 1 over $A$ from the Lie algebra $W_n(\mathbb{K})$. The main result, Theorem 11, states that such a Lie algebra $L$ is determined by a reduced derivation $D \in W_n(\mathbb{K})$ and elements $\lambda,\ a,\ b \in A$ such that

$$D(a) = \lambda a, D(b) = \lambda b + c \text{ for some } c \in \mathrm{Ker}\, D.$$

The set

$$VD,\ V \subseteq A^D_\lambda,$$

forms an abelian ideal $I$ of the Lie algebra $L$ with the abelian factor algebra $L/I$. The obtained description can be useful for studying solvable Lie algebras of small rank over $A$.

We use standard notations. A nonzero polynomial $a \in A$ is called *a Darboux polynomial* for a derivation $D \in W_n(\mathbb{K})$ if $D(a) = \lambda a$ for some $\lambda \in A$. Such a polynomial $\lambda$ is called *a cofactor* for $D$ with respect to $a$. Some properties of Darboux polynomials and their applications in theory of differential equations can be found in [7; 8]. We denote by $A^D_\lambda$ the set of all Darboux polynomials for $D \in W_n(\mathbb{K})$ with the same cofactor $\lambda$ and of the zero polynomial. The set $A^D_\lambda$ is obvioulsy a vector space over $\mathbb{K}$. If $V$ is a subspace of $A^D_\lambda$, then we denote by $VD$ the set of all derivations $fD$, $f \in V$. This set is a vector space over the field $\mathbb{K}$. For polynomials $f,\ g \in A$, we denote by $\gcd(f, g)$ the greatest common divisor of $f$ and $g$.

## Abelian Lie algebras of rank one

Some auxiliary results are collected in the next three lemmas (see, for example [6], [8]).

**Lemma 1.** *Let $D_1,\ D_2 \in W_n(\mathbb{K})$ and $a,\ b \in A$. Then*

1. $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$.
2. *If* $a, b \in \operatorname{Ker} D_1 \cap \operatorname{Ker} D_2$, *then* $[aD_1, bD_2] = = ab[D_1, D_2]$.
3. *If* $[D_1, D_2] = 0$, *then* $[aD_1, bD_2] = = aD_1(b)D_2 - bD_2(a)D_1$.

**Lemma 2.** *Let $L$ be an abelian subalgebra of rank one over $A$ of the Lie algebra $W_n(\mathbb{K})$ and $aD$, $bD \in L$ for some nonzero $D \in W_n(\mathbb{K})$, $a, b \in \in A$, $b \neq 0$. Then $D(\frac{a}{b}) = 0$. Moreover, if $h = = \gcd(a, b)$ and $a = a_1 h$, $b = b_1 h$ for some $a_1, b_1 \in \in A$, then $[a_1 D, b_1 D] = 0$.*

*Proof.* Using Lemma 1, one can get

$$[aD, bD] = (aD(b) - bD(a))D = 0,$$

since $L$ is abelian and $aD$, $bD \in L$. Then $aD(b) - bD(a) = 0$. This implies

$$D\left(\frac{a}{b}\right) = \frac{D(a)b - aD(b)}{b^2} = 0.$$

If $h = \gcd(a, b)$ and $a = a_1 h$, $b = b_1 h$, then

$$D(\frac{a}{b}) = D(\frac{a_1}{b_1}) = 0$$

The last equation implies the equality $[a_1 D, b_1 D] = 0$. The proof is complete.

We need the following properties of Darboux polynomials.

**Lemma 3.** *Let $D \in W_n(\mathbb{K})$. Then*

1. *If $f$, $g \in A$ are Darboux polynomials with cofactors $\lambda$ and $\mu$ respectively, then $fg$ is a Darboux polynomial for $D$ with the cofactor $\lambda + \mu$. Furthermore, if $g \neq 0$, then*
$$D\left(\frac{f}{g}\right) = (\lambda - \mu)\frac{f}{g}.$$
2. *Every irreducible divisor of a Darboux polynomial for $D$ is again a Darboux polynomial for $D$.*
3. *If $f$ and $g \neq 0$ are coprime polynomials and $D(f/g) = 0$, then $f$ and $g$ are Darboux polynomials with the same cofactor.*

*Proof.* See, for example, [7] and Propositions 2.2.1, 2.2.2 in [8].

**Lemma 4.** *Let $D \in W_n(\mathbb{K})$. The set $A_0 \subseteq A$ of all Darboux polynomials for the derivation $D$ with all possible cofactors is closed under multiplication. The set $A_1 \subseteq A$ of all polynomials that are not divisible by any non-constant Darboux polynomial for $D$ has the same property. Moreover, the equality $A = A_0 \cdot A_1$ holds.*

*Proof.* The multiplicative closure of the set $A_0$ follows from Lemma 3(1). Let $f$, $g \in A_1$ and suppose some Darboux polynomial $h$ divides $fg$. Then by Lemma 3(3) we may assume without lost of generality that $h$ is irreducible. Since $A$ is a unique factorization domain, we get that $h$ divides either $f$ or $g$, which is impossible. Thus $A_1 A_1 \subseteq A_1$. The equality $A = A_0 A_1$ follows from the multiplicative closeness of sets $A_0$, $A_1$ and the fact that $A$ is a UFD. The proof is complete.

**Corollary 5.** *If $f \in A, f \notin \mathbb{K}$, then $f = f_0 f_1$ for the uniquely determined polynomials $f_0 \in A_0$ and $f_1 \in A_1$.*

**Definition.** A derivation $D$ of the polynomial ring $A$ is called a *reduced derivation* if the condition $D = fD_1$ with $D_1 \in W_n(\mathbb{K})$ and $f \in A$ implies that $f$ is a constant polynomial.

We will use the next statement, which can be found in [1].

**Lemma 6.** *[1, Lemma2] For every submodule $M$ of the $A$-module $W_n(\mathbb{K})$ of rank one over $A$ there exist an ideal $I$ of $A$ and a reduced derivation $D_0 \in \in W_n(\mathbb{K})$ such that $M = ID_0$. The submodule $M$ defines the derivation $D_0$ uniquely up to a nonzero scalar.*

**Theorem 7.** *Let $L$ be an abelian subalgebra of rank one over $A$ of the Lie algebra $W_n(\mathbb{K})$ and $\dim_{\mathbb{K}} L \geq 2$. Then there exist elements $D \in W_n(\mathbb{K})$ and $\lambda \in A$ such that $L = VD$ for some $\mathbb{K}$-subspace $V$ of $A^D_\lambda = \{a \in A \mid D(a) = \lambda a\}$. Conversely, every subalgebra of $W_n(\mathbb{K})$ of such a form is abelian of rank one over $A$.*

*Proof.* Since $W_n(\mathbb{K})$ is a free $A$-module and a subalgebra $L \subseteq W_n(\mathbb{K})$ is of rank 1 over $A$, by Lemma 6 the subalgebra $AL \subseteq W_n(\mathbb{K})$ is of the the form $AL = ID$ for an ideal $I$ of the ring $A$ and some reduced derivation $D \in W_n(\mathbb{K})$ (note that in general $D \notin L$). The inclusion $L \subseteq AL$ implies that every element of the subalgebra $L$ is of the form $aD$ for some $a \in A$ and $D \in W_n(\mathbb{K})$. It is easy to see that the set

$$U = \{a \in A | aD \in L\}$$

is a vector subspace (over $\mathbb{K}$) of the algebra $A$. Let us choose a basis $\{\bar{a}_i, i \in J\}$ of the vector space $U$ over $\mathbb{K}$, where $J$ is a set of indices, and denote by $h$ the greatest common divisor of elements $\overline{a}_i \in \in A$, $i \in J$. Then $\overline{a}_i = ha_i$ for some $a_i \in A$, $i \in J$, and $U = hV$, where $V$ is a vector space over $\mathbb{K}$ with a basis $\{a_i, i \in J\}$ such that $\gcd_{i \in J} a_i = 1$. Hence, we get

$$L = hV \cdot D.$$

Let us fix an arbitrary element $b \in V$ and show that $b$ is a Darboux polynomial for the derivation $D$. By definition of the vector space $V$, we get $hbD \in L$. Then for each basic element $a_i, i \in J$, the relation $[hbD, ha_i D] = 0$ holds. We therefore have $D(b/a_i) = 0$ by Lemma 2. Denote $d_i = \gcd(b, a_i)$, $i \in J$. Then

$$a_i = c_i d_i, \ b = \overline{b}_i d_i$$

for some coprime polynomials $\bar{b}_i$ and $c_i$, $i \in J$. Since $D(\bar{b}_i/c_i) = 0$, Lemma 3(3) implies that $\bar{b}_i$ and $c_i$ are Darboux polynomials for $D$. By Lemma 3(2), each irreducible divisor of the polynomial $c_i$, $i \in J$, is a Darboux polynomial for the derivation $D$.

Let us show that each irreducible divisor of the polynomial $b$ is a divisor of at least one of polynomials $\bar{b}_i$, $i \in J$. Suppose to the contrary that there is an irreducible polynomial $r$ such that $r \mid b$ and $r \nmid \bar{b}_i$ for all $i \in J$. But then $r \mid d_i$ for all $i \in J$. One can easily show that the last relations imply $r \mid a_i, i \in J$. The latter is impossible because $\gcd_{i\in J} a_i = 1$. The obtained contradiction shows that every divisor of the polynomial $b$ is a divisor of at least one of polynomials $\bar{b}_i$, $i \in J$. Then by Lemma 3(2) each divisor of $b$ is a Darboux polynomial for $D$ and thus, $b$ also is a Darboux polynomial for $D$.

Let now $b_1, b_2 \in V$ be arbitrary elements. Then

$$D(b_1) = \lambda_1 b_1,\ D(b_2) = \lambda_2 b_2$$

for some $\lambda_1$, $\lambda_2 \in A$ as noted above. Since $[hb_1D, hb_2D] = 0$ we get $\lambda_1 = \lambda_2$ and then $b_1, b_2 \in A_\lambda^D$, where $\lambda = \lambda_1 = \lambda_2$. It follows from the last relation that $V \subseteq A_\lambda^D$. Denoting by $D$ the derivation $hD$, we obtain $L = VD$.

The converse statement can be proven by a straightforward check.

*Remark* 1. It follows from the proof of Theorem 7 that under conditions of the theorem the Lie algebra $L$ can be written in the form $L = hVD$ for some reduced derivation $D \in W_n(\mathbb{K})$, a polynomial $h \in A$, and a subspace $V \subseteq A_\lambda^D$.

## Nonabelian solvable Lie algebras of derivations of rank 1

**Lemma 8.** *[6, Lemma 7] Let $L$ be a solvable nonabelian subalgebra of rank 1 over $A$ from the Lie algebra $W_n(\mathbb{K})$. Then the derived length $s(L) = 2$.*

**Lemma 9.** *Let $L$ be a solvable nonabelian subalgebra of $W_n(\mathbb{K})$ of rank 1 over $A$. Then $L$ contains a maximal (with respect to inclusion) abelian ideal $I$ of the form $I = hVD$ for a derivation $D \in W_n(\mathbb{K})$ and a subspace $V \subseteq A_\lambda^D$, $\lambda \in A$. Moreover, each element from $L \setminus I$ is of the form $bD$, where $b \in A$, and $[bD, ahD] = \mu ahD$ for some $\mu = \mu(b) \in \operatorname{Ker} D$.*

*Proof.* By Lemma 8, the Lie algebra $L$ is solvable with the derived length $s(L) = 2$. Let $I$ be a maximal abelian ideal of $L$ that contains the abelian ideal $L' = [L, L]$. Then $I = hVD$ for a reduced derivation $D \in W_n(\mathbb{K})$ and a subspace $V \subseteq A_\lambda^D$ by Theorem 7. Since the set of Darboux polynomials for $D$ is multiplicatively closed (Lemma 2), we may assume without lost of generality that $h$ is not divisible by any non-constant Darboux polynomial for $D$.

Let $ahD \in I$, $a \in V$, $a \neq 0$, and $bD \in L \setminus I$, $b \in A$ be arbitrary elements. Then

$$[bD, ahD] = (bD(ah) - ahD(b))D \in I.$$

Since $a \in V$, we have $D(a) = \lambda a$ by definition of the set $V$. Therefore

$$bD(ah) - ahD(b) = \lambda abh + abD(h) - ahD(b) \in hV$$

and we obtain

$$\lambda abh + abD(h) - ahD(b) = \bar{a}h \tag{1}$$

for some polynomial $\bar{a} \in V$.

Since $h$ is not divisible by any non-constant Darboux polynomial for $D$, it follows from the equation (1) that each divisor of $a$ divides the polynomial $\bar{a}$. Then $a$ divides $\bar{a}$ and $\bar{a} = \mu a$ for some polynomial $\mu \in A$. We have $D(\bar{a}/a) = 0$, since $a$, $\bar{a} \in V \subseteq A_\lambda^D$. Therefore $\mu = \bar{a}/a \in \operatorname{Ker} D$ and we get the relation

$$\lambda abh + abD(h) - ahD(b) = \mu ah. \tag{2}$$

The latter means that

$$[bD, ahD] = \mu ahD \tag{3}$$

for some $\mu \in \operatorname{Ker} D$.

Let us show that the element $\mu \in \operatorname{Ker} D$ depends only on the element $bD \in L \setminus I$ and doesn't depend on $ahD \in I$.

Take arbitrary elements $a_1hD$, $a_2hD \in I$ and denote

$$[bD, a_1hD] = \mu_1 a_1 hD, \tag{4}$$

$$[bD, a_2hD] = \mu_2 a_2 hD \tag{5}$$

for $\mu_i \in \operatorname{Ker} D$, $i = 1, 2$. We consider two cases.

Firstly, let $\operatorname{rk}_{\operatorname{Ker} D} I = 1$. Then for $a_1hD$, $a_2hD \in I$ there exist nonzero elements $\nu_1, \nu_2 \in \operatorname{Ker} D$ such that

$$\nu_1 a_1 hD + \nu_2 a_2 hD = 0. \tag{6}$$

Thus

$$a_1hD = -\frac{\nu_2}{\nu_1} a_2 hD$$

and obviously $D(\nu_2/\nu_1) = 0$. Therefore

$$[bD, a_1hD] = [bD, -\frac{\nu_2}{\nu_1} a_2hD] = $$

$$= -\frac{\nu_2}{\nu_1}[bD, a_2hD]. \tag{7}$$

Using equalities (4), (5) and (7), we get

$$[bD, a_1hD] = -\frac{\nu_2}{\nu_1}\mu_2 a_2 hD = \mu_1 a_1 hD. \tag{8}$$

From the last relations we obtain

$$-(\nu_2/\nu_1)\mu_2 a_2 h = \mu_1 a_1 h$$

and thus

$$\mu_1\nu_1 a_1 + \mu_2\nu_2 a_2 = 0.$$

By (6) $\nu_1 a_1 = -\nu_2 a_2$, so we have

$$\mu_2\nu_2 a_2 - \mu_1\nu_2 a_2 = 0,$$

and thus $\mu_1 = \mu_2$. Therefore,

$$[bD, ahD] = \mu \cdot ahD$$

for an arbitrary $a \in V$, and $\mu = \mu(b) \in \operatorname{Ker} D$ depends only on $b$.

Now, let $\operatorname{rk}_{\operatorname{Ker} D} I > 1$ and elements $a_1 hD,\ a_2 hD \in I$ be linearly independent over $\operatorname{Ker} D$. We use notations (4), (5) from the above considered. From the relation (2) we get

$$\begin{aligned} \lambda a_1 bf + a_1 bD(h) - a_1 hD(b) &= \mu_1 a_1 h, \\ \lambda a_2 bf + a_2 bD(h) - a_2 hD(b) &= \mu_2 a_2 h. \end{aligned} \tag{9}$$

Furthermore, for an element $(a_1 + a_2)hD \in I$ we have

$$[bD, (a_1 + a_2)hD] = \nu(a_1 + a_2)hD \tag{10}$$

for some $\nu \in \operatorname{Ker} D$. It follows from (9) that

$$[bD, (a_1 + a_2)hD] = (\mu_1 a_1 + \mu_2 a_2)hD. \tag{11}$$

Using (10) and (11), we obtain

$$\mu_1 a_1 + \mu_2 a_2 = \nu(a_1 + a_2).$$

Since $a_1,\ a_2$ are linearly independent over $\operatorname{Ker} D$, it follows from the last relation $\mu_1 = \mu_2 = \nu$. As in the case of $\operatorname{rk}_{\operatorname{Ker} D} I = 1$, it is easy to see that

$$[bD, ahD] = \mu ahD$$

for the same $\mu$ and an arbitrary element $ahD \in I$. The proof is complete.

**Corollary 10.** *Under conditions of the lemma, if* $\operatorname{Ker} D = \mathbb{K}$, *then* $\operatorname{ad} bD$ *acts as a scalar linear operator on the ideal* $I$.

**Theorem 11.** *Let $L$ be a solvable nonabelian subalgebra of the Lie algebra $W_n(\mathbb{K})$ with $\operatorname{rk}_A L = 1$. Then $L$ contains an abelian ideal $I$ of the form $I = VhD$ for a derivation $D \in W_n(\mathbb{K})$, a polynomial $h \in A$, a subspace $V \subseteq A^D_\lambda$ and each element from $L \setminus I$ is of the form $bD$ for $b \in A$ such that $D(b) = \lambda b + c$ for some $c \in \operatorname{Ker} D$. Moreover, $[bD, ahD] = c \cdot ahD$ for an arbitrary $ahD \in I$.*

*Proof.* In view of Lemma 8, $L$ is solvable of derived length $s(L) = 2$. Therefore, $L' = [L, L]$ is an abelian ideal. Let us denote by $I$ any maximal abelian ideal of $L$ that contains $L'$. Then the centralizer $C_L(I) = I$ and $L/I$ is an abelian quotient algebra. By Theorem 7 $I = hVD$ for a reduced derivation $D \in W_n(\mathbb{K})$, a subspace $V \subseteq A^D_\lambda$, and a polynomial $h \in A$. Without lost of generality, we may assume that $h$ is not divisible by any non-constant Darboux polynomial for $D$.

Let us choose an arbitrary element $bD \in L \setminus I$. By Lemma 9,

$$[bD, ahD] = \mu ahD$$

for some $\mu = \mu(b) \in \operatorname{Ker} D$. As in Lemma 9 it is easy to show that

$$\lambda bh + bD(h) - hD(b) = \mu h. \tag{12}$$

It follows from the equation (12) that $\gcd(b, h) \neq 1$. Indeed, otherwise, $D(h)$ is divisible by $h$ and $h$ is a Darboux polynomial for $D$, which contradicts our assumption. Thus, we may choose an element $b_0 D \in L \setminus I$ with the highest degree $\deg \gcd(b_0, h)$. Denote $h_0 = \gcd(b_0, h)$, $h_0 \neq \mathrm{const}$.

Let us show that $b$ is divisible by $h_0$ and $b/h_0$ is coprime with $h$ for an arbitrary element $bD \in L \setminus I$. By Lemma 9, we have

$$[bD, ahD] = \mu ahD, \quad [b_0 D, ahD] = \mu_0 ahD$$

for an arbitrary $ahD \in I$ and some $\mu, \mu_0 \in \operatorname{Ker} D$. It is obvious that

$$[\mu_0 bD - \mu b_0 D, ahD] = 0$$

for an arbitrary $ahD \in I$. Since $C_L(I) = I$, we get

$$\mu_0 bD - \mu b_0 D \in I,$$

that is

$$\mu_0 b - \mu b_0 = a_0 h \tag{13}$$

for some $a_0 \in V$. Since $b_0$ is divisible by $h_0 = \gcd(b_0, h)$, it follows from (13) that $\mu_0 b$ is divisible by $h_0$. Note that $\mu_0$ and $h_0$ are coprime polynomials. Indeed, suppose to the contrary that $\overline{h}_0 = \gcd(\mu_0, h_0)$ is a non-constant polynomial. Then $\overline{h}_0$ is a Darboux polynomial for $D$ since $\overline{h}_0$ divides $\mu_0 \in \operatorname{Ker} D$. But $\overline{h}_0$ divides $h_0$ and $h_0$ divides $h$, so $\overline{h}_0$ divides $h$, which contradicts our assumption on the element $h$.

Relations (13) and $\gcd(\mu_0, h_0) = 1$ imply that $b$ is divisible by $h_0$. Hence, it is easy to see that $b/h_0$ is a coprime polynomial to $h$ for an arbitrary $bD \in L \setminus I$.

Let us denote $\overline{b} = b/h_0$. Then $bD = \overline{b} h_0 D$ for an arbitrary $bD \in L \setminus I$. We (for convenience) use the following notations

$$D_1 = h_0 D,\ h_1 = \frac{h}{h_0},\ \lambda_1 = h_0 \lambda.$$

Then

$$I = hVD = h_1 h_0 VD = h_1 V_1 D_1,$$

where $V_1 \subseteq A_{\lambda_1}^{D_1}$. Moreover, each element from $L \backslash I$ may be presented in the form $b_1 D_1$ for $b_1 \in A$ and $D_1 = h_0 D$. Note that without lost of generality we may choose an element $h_1 = h/h_0$ such that $h_1$ is not divisible by any non-constant Darboux polynomial for $D_1$. As in Lemma 9, one can show that for an arbitrary element $a_1 h_1 D_1 \in I$ there exists $\mu \in \operatorname{Ker} D_1$ (note that $\operatorname{Ker} D_1 = \operatorname{Ker} D$) such that

$$[b_1 D_1, a_1 h_1 D_1] = \mu a_1 h_1 D_1.$$

Then the equality

$$\lambda_1 b_1 h_1 + b_1 D_1(h_1) - h_1 D_1(b_1) = \mu h_1 \tag{14}$$

holds and this equality implies that $b_1 D_1(h_1)$ is divisible by $h_1$. By the proved above

$$\gcd(b_1, h) = 1 = \gcd(b_1, h_1).$$

These equalities imply that $D_1(h_1)$ is divisible by $h_1$, whence $h_1$ is a Darboux polynomial for $D_1$. The last contradicts our choice of $h_1$ and thus $h_1 = \text{const}$. Then it follows from (14) that $\lambda_1 b_1 - D(b_1) = \mu$ for some $\mu \in \operatorname{Ker} D_1$, that is

$$D(b_1) = \lambda_1 b_1 + \mu, \ \mu \in \operatorname{Ker} D_1.$$

Replacing notations, we obtain the statement of the theorem.

*Example* 1. Let $I$ and $B$ be vector spaces of derivations of $W_2(\mathbb{K})$ of the following forms:

$$I = \mathbb{K}\langle x_2 \tfrac{\partial}{\partial x_1}, {x_2}^2 \tfrac{\partial}{\partial x_1}, \dots {x_2}^m \tfrac{\partial}{\partial x_1}, \dots \rangle$$

$$B = \mathbb{K}\langle x_2 x_1 \tfrac{\partial}{\partial x_1}, {x_2}^2 x_1 \tfrac{\partial}{\partial x_1}, \dots {x_2}^m x_1 \tfrac{\partial}{\partial x_1}, \dots \rangle$$

Then one can easily check that $I$ and $B$ are abelian subalgebras of $W_2(\mathbb{K})$ and $[B, I] \subseteq I$. Thus, $L = B + I$ is a metabelian Lie algebra of rank 1 over $A$. This Lie algebra is of type described in Theorem 11 when we put $D = \frac{\partial}{\partial x_1}, \lambda = 0, h = 1$.

### *References*


1. I. V. Arzhantsev, E. A. Makedonskii, A. P. Petravchuk, “Finite-dimensional subalgebras in polynomial Lie algebras of rank one”, Ukrainian Math. Journal. **63** (5), 827–832 (2011).
2. A. Cohen and J. Draisma, “From Lie algebras of vector fields to algebraic group actions”, Transformation Groups. (8), 51–68 (2003).
3. A. González-López, N. Kamran, and P. J. Olver, “Lie algebras of differential operators in two complex variables”, Amer. J. Math. **114**, 1163–1185 (1992).
4. A. González-López, N. Kamran, and P. J. Olver, “Lie algebras of vector fields in the real plane”, Proc. London Math. Soc. **64** (3), 339–368 (1992).
5. S. Lie, *Theorie der Transformationsgruppen, Vol. 3* (Teubner, Leipzig, 1893).
6. Ie. O. Makedonskyi and A. P. Petravchuk, “On nilpotent and solvable Lie algebras of derivations”, Journal of Algebra. **401**, 245–257 (2014).
7. J. M. Ollagnier, “Algebraic closure of a rational function”, Qualitative Theory of Dynamical Systems. **5** (2), 285–297 (2004).
8. A. Nowicki, *Polynomial Derivations and their Rings of Constants* (Uniwersytet Mikolaja Kopernika, Torun, 1994).


*Петравчук А. П., Сисак К. Я.*

# РОЗВ’ЯЗНІ АЛГЕБРИ ЛІ ДИФЕРЕНЦІЮВАНЬ РАНГУ ОДИН

Нехай $\mathbb{K}$ – довільне поле характеристики нуль, $A = \mathbb{K}[x_1, \dots, x_n]$ – кільце многочленів та $R = \mathbb{K}(x_1, \dots, x_n)$ – поле раціональних функцій від $n$ змінних над $\mathbb{K}$. Алгебра Лі $W_n(\mathbb{K})$ всіх $\mathbb{K}$-диференціювань кільця $A$ становить великий інтерес, оскільки її елементи можуть розглядатися як векторні поля на $\mathbb{K}^n$ з поліноміальними коефіцієнтами. Якщо $L$ підалгебра із $W_n(\mathbb{K})$, то можна визначити ранг $\operatorname{rk}_A L$ підалгебри $L$ над кільцем $A$ як розмірність векторного простору $RL$ над полем $R$. Скінченновимірні (над $\mathbb{K}$) підалгебри рангу 1 над $A$ вивчалися першим автором разом з І. Аржанцевим та Є. Македонським. Ми вивчаємо розв’язні підалгебри $L$ алгебри Лі $W_n(\mathbb{K})$ з $\operatorname{rk}_A L = 1$, без обмежень на розмірність над $\mathbb{K}$. Дано опис таких алгебр Лі в термінах многочленів Дарбу.

**Ключові слова:** алгебра Лі, розв’язна алгебра Лі, диференціювання, многочлен Дарбу, кільце многочленів.

*Матеріал надійшов 20.07.2019*