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COMPUTING THE MOORE-PENROSE INVERSE FOR BIDIAGONAL MATRICES

The Moore-Penrose inverse is the most popular type of matrix generalized inverses which has many applications both in matrix theory and numerical linear algebra. It is well known that the Moore-Penrose inverse can be found via singular value decomposition. In this regard, there is the most effective algorithm which consists of two stages. In the first stage, through the use of the Householder reflections, an initial matrix is reduced to the upper bidiagonal form (the Golub-Kahan bidiagonalization algorithm). The second stage is known in scientific literature as the Golub-Reinsch algorithm. This is an iterative procedure which with the help of the Givens rotations generates a sequence of bidiagonal matrices converging to a diagonal form. This allows to obtain an iterative approximation to the singular value decomposition of the bidiagonal matrix.

The principal intention of the present paper is to develop a method which can be considered as an alternative to the Golub-Reinsch iterative algorithm. Realizing the approach proposed in the study, the following two main results have been achieved. First, we obtain explicit expressions for the entries of the Moore-Penrose inverse of bidigonal matrices. Secondly, based on the closed form formulas, we get a finite recursive numerical algorithm of optimal computational complexity. Thus, we can compute the Moore-Penrose inverse of bidiagonal matrices without using the singular value decomposition.

Keywords: Moor-Penrose inverse, bidiagonal matrix, inversion formula, finite recursive algorithm.

Introduction

As is known, for a real $m \times n$ matrix A the Moore-Penrose inverse A^+ is the unique matrix that satisfies the following four properties [1]:

$$AA^+A = A\,,\; A^+AA^+ = A^+\,,$$

$$(A^+A)^T = A^+A\,,\; (AA^+)^T = AA^+\,.$$

If A is a square nonsingular matrix, then $A^+ = A^{-1}$. Thus the Moore-Penrose inverse generalizes the ordinary matrix inversion.

There is well-known formula for the Moore-Penrose inverse which is obtained by the *singular value decomposition* (abbreviated SVD) of the matrix (see [1; 4], for instance).

The singular value decomposition of an $m \times n$ matrix A with rank r is its factorization of the form

$$A = U\Lambda V^T, \tag{1.1}$$

where U is an $m \times m$ orthogonal matrix, $\Lambda = diag\,[\sigma_1, \sigma_2, \ldots, \sigma_r]$ is an $m \times n$ diagonal matrix, and V is an $n \times n$ orthogonal matrix. The diagonal entries $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ of Λ are known as *singular values* of the matrix A. Having the factorization (1.1), the Moore-Penrose inverse can be written as

$$A^+ = V\Lambda^+U^T, \tag{1.2}$$

where $\Lambda^+ = diag\,[\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_r^{-1}]$ is $n \times m$ diagonal matrix.

The most effective procedure to compute the Moore-Penrose inverse involves two main stages [4].

Stage 1. Matrix reduction to the bidiagonal form.

At this stage an $m \times n$ matrix, where $m \geq n$, by means of the Householder reflections is transformed to upper bidiagonal form

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & 0 & \ldots & 0 \\ 0 & a_{22} & a_{23} & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & a_{n-1\,n-1} & a_{n-1\,n} \\ 0 & \ldots & 0 & 0 & a_{nn} \\ \hline & & \mathbf{0} & & \end{array}\right]. \tag{1.3}$$

The computational process is known as *Golub-Kahan bidiagonalization* [2]. Thereby the problem is reduces to the Moore-Penrose inversion of the bidiagonal matrix (1.3).

Stage 2. Golub-Reinsch SVD iterative algorithm.

Once the bidiagonalization of the initial matrix has been achieved, the next task is to zero the superdiagonal entries in the matrix (1.3). With this purpose the *Golub-Reinsch algorithm* is implemented [3]. The algorithm, with the help of the Givens rotations generates a sequence of bidiagonal matrices that converge to a diagonal form. As a result, at a certain step of the iterative process we get an approximation to the SVD of the bidiagonal matrix (1.3). Having the SVD, the Moore-Penrose inverse of the matrix is computed (see [1; 4], for instance).

The objective of the present work is to develop a method which allows to deduce formulas for the entries of the Moore-Penrose inverse of upper bidiagonal matrices. The obtained closed form solution to the Moore-Penrose inversion may be considered as an alternative to sufficiently labour-consuming Golub-Reinsch iterative procedure briefly described in the Stage 2 of this section. Moreover, explicit expressions for the entries of the Moore-Penrose inverse lead to fairly simple finite numerical algorithm with optimal volume of computational expenditures.

Partition of a bidiagonal matrix into blocks

Let us consider a real $n \times n$ upper bidiagonal matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & & & \\ & a_{22} & a_{23} & 0 & \\ & & \ddots & \ddots & \\ & 0 & & a_{n-1\,n-1} & a_{n-1\,n} \\ & & & & a_{nn} \end{bmatrix}. \qquad (2.1)$$

Note that it suffices to consider square upper bidiagonal matrices since for rectangular upper bidiagonal matrices the problem can be easily reduced to our case. Indeed, if $m > n$ then according to (1.3) we have the block structure

$$\begin{bmatrix} A \\ 0 \end{bmatrix},$$

where A is a square upper bidiagonal matrix of the form (2.1). It can be seen that in this case

$$\begin{bmatrix} A \\ 0 \end{bmatrix}^+ = \begin{bmatrix} A^+ & 0^T \end{bmatrix}.$$

We assume that the matrix A is singular, i.e. $a_{11}a_{22}\dots a_{nn} = 0$. Next, we assume that

$$a_{i\,i+1} \neq 0,\ i = 1, 2, \dots, n-1. \qquad (2.2)$$

Otherwise, if some of superdiagonal entries of the matrix A are equal to zero, the problem of computing the Moore-Penrose inverse is decomposed into several similar problems for bidiagonal matrices of lower order.

To compute the Moore-Penrose inverse of the matrix A, we apply a special partition of this matrix into blocks. The partitioning procedure uses the arrangement of zeros on the main diagonal of the matrix. We distinguish the following four cases.

Case 1: $a_{11} \neq 0$, $a_{nn} \neq 0$.

Let zero diagonal entries of the matrix A are $a_{i_1 i_1}, a_{i_2 i_2}, \dots, a_{i_{p-1} i_{p-1}}$, where $1 < i_1 < i_2 < \dots < i_{p-1} < n$ and $p > 1$. We split the matrix into blocks drawing dividing lines after the rows $i_1 - 1, i_2 - 1, \dots, i_{p-1} - 1$ and after the columns $i_1, i_2, \dots, i_{p-1}$. As a result, the matrix (2.1) takes a block diagonal form. The first and the last diagonal blocks are rectangular bidiagonal matrices of the sizes $(i_1 - 1) \times i_1$ and $(n - i_{p-1} + 1) \times (n - i_{p-1})$, respectively, while the remaining blocks are square lower bidiagonal matrices. As an illustration, a pattern of the matrix (for $n = 10$), the partitioning procedure and resulting block diagonal structure are shown in Fig. 2.1 (stars represent nonzero entries).

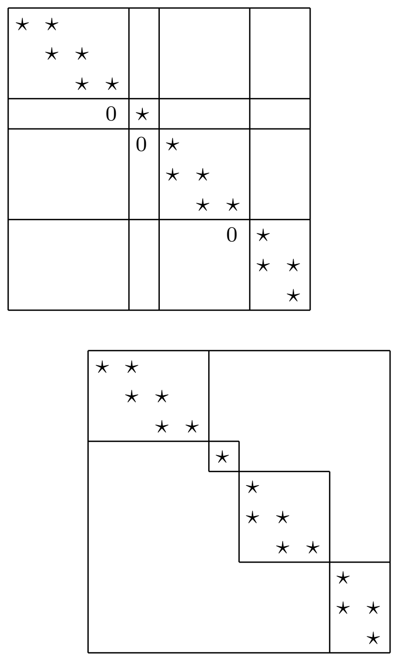


Figure 2.1. Partition of the matrix (case 1).

Case 2: $a_{11} = 0$, $a_{nn} \neq 0$.

We allocate the first column of the matrix A, as a separate zero block of the size $n \times 1$. Next, we partition the remaining submatrix into diagonal blocks as follows. If there are other zero diagonal entries of the matrix A, say $a_{i_1 i_1}, a_{i_2 i_2}, \dots, a_{i_{p-1} i_{p-1}}$, where $1 < i_1 < i_2 < \dots < i_{p-1} < n$ and $p > 1$, then the submatrix is subdivided according to the rule drscribed in the Case 1. As an illustration, see a pattern of the matrix given in Fig. 2.2 . The last diagonal block of the submatrix is rectangular bidiagonal matrix of the size $(n - i_{p-1} + 1) \times (n - i_{p-1})$; the remaining diagonal blocks are square lower bidiagonal matrices. If there are no other zero diagonal entries, except the first one, then the submatrix is not subdivided.

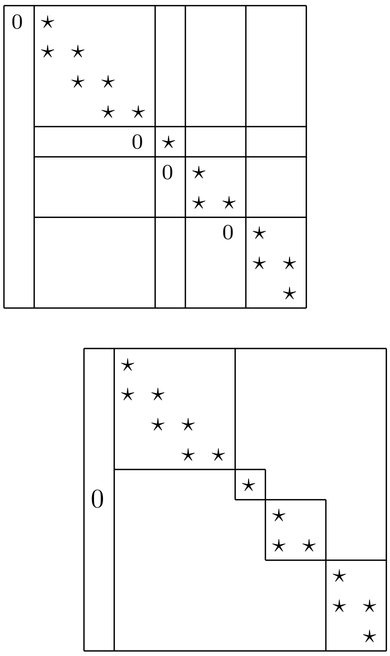
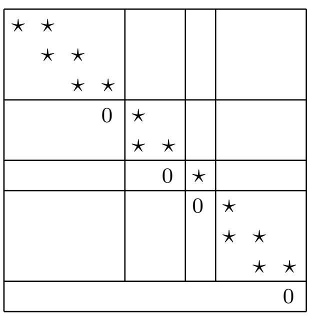


Figure 2.2. Partition of the matrix (case 2).

Case 3: $a_{11} \neq 0$, $a_{nn} = 0$.

First, we allocate the last row of the matrix A, as a separate zero block of the size $1 \times n$. Next, we partition the remaining submatrix into diagonal blocks using the same idea. If there are other zero diagonal entries of the matrix A, say $a_{i_1 i_1}, a_{i_2 i_2}, \ldots, a_{i_{p-1} i_{p-1}}$, where $1 < i_1 < i_2 < \cdots < i_{p-1} < n$ and $p > 1$, then the submatrix is subdivided by the rule drscribed in the Case 1 (see a pattern of the matrix given in Fig. 2.3). The first diagonal block of the submatrix is rectangular bidiagonal matrix of the size $(i_1 - 1) \times i_1$; the remaining diagonal blocks are square lower bidiagonal matrices. If there are no other zero diagonal entries, except the last one, then the submatrix is not subdivided.



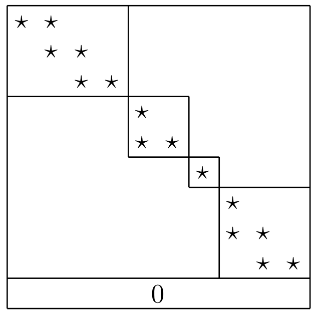


Figure 2.3. Partition of the matrix (case 3).

Case 4: $a_{11} = 0$, $a_{nn} = 0$.

The allocation of the first column and the last row of the matrix A gives us three zero blocks of the sizes $(n-1) \times 1$, $1 \times (n-1)$ and 1×1 (see Fig. 2.4). Then we partition the remaining submatrix. If there are other zero diagonal entries of the matrix A, say $a_{i_1 i_1}, a_{i_2 i_2}, \ldots, a_{i_{p-1} i_{p-1}}$, where $1 < i_1 < i_2 < \cdots < i_{p-1} < n$ and $p > 1$, then the submatrix is subdivided by the rule drscribed in the Case 1. The diagonal blocks of this subdivision are square lower bidiagonal matrices. If there are no other zero diagonal entries except the first and last, then the submatrix is not subdivided.

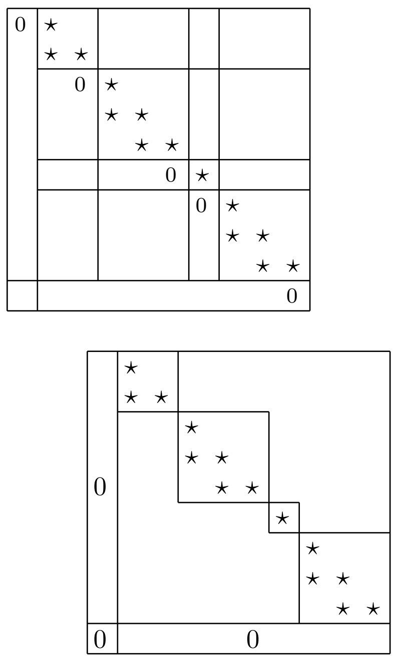


Figure 2.4. Partition of the matrix (case 4).

Thus we have four principal cases of block partitioning the initial upper bidiagonal matrix A, schematically presented in Fig. 2.5.

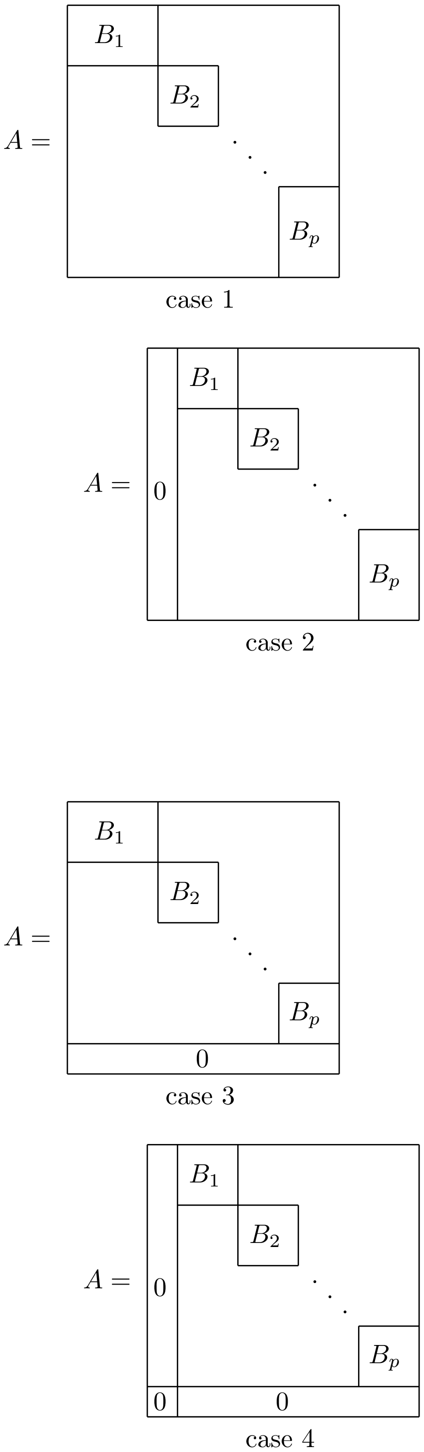


Figure 2.5. The cases of block partitioning.

Accordingly, the Moore-Penrose inverse also has a block structure, as shown in Fig. 2.6.

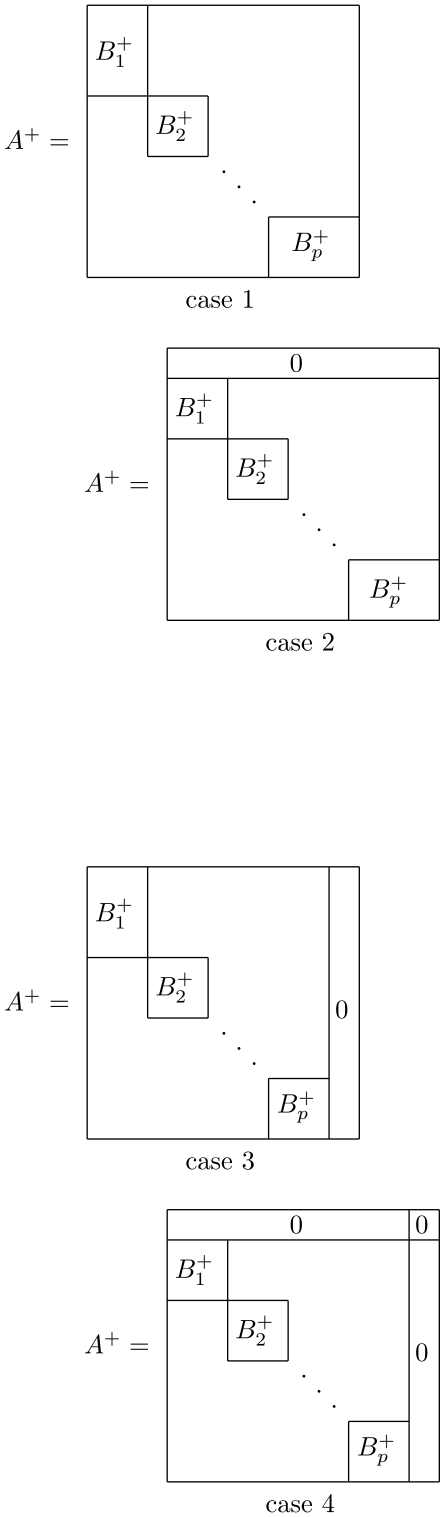


Figure 2.6. The structure of the Moore-Penrose inverse.

Summarizing the previous reasoning, we conclude that our task is to find the Moore-Penrose inverses for blocks of the following three types:

type 1 : bidiagonal block of a size $m \times m$;

type 2 : bidiagonal block of a size $m-1 \times m$;

type 3 : bidiagonal block of a size $m \times m-1$.

In Fig. 2.7 we schematically give the types of diagonal blocks (the mark $\star$ stands for a nonzero entry).

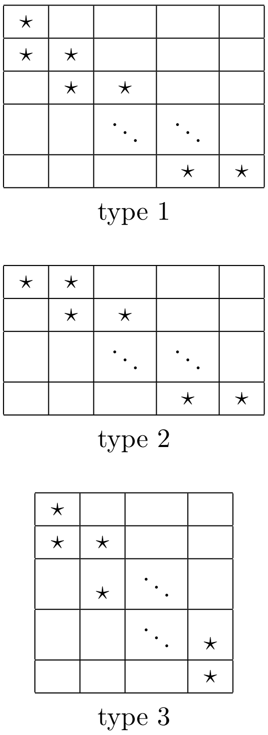


Figure 2.7. The types of diagonal blockfs.

It is necessary to pay attention to the following circumstance. As follows from the process of partitioning the initial matrix (2.1) into blocks, in each of the Cases 1-4 we have at most two rectangular blocks (of size $m-1 \times m$ or $m \times m-1$). The remaining blocks are square lower bidiagonal matrices. As an illustration, see Fig. 2.5.

Computing the Moore-Penrose inverse for a block of the type 1 is not difficult. Consider a square matrix

$$B = \begin{bmatrix} d_1 & & & \\ b_1 & d_2 & & \\ & \ddots & \ddots & \\ & & b_{m-1} & d_m \end{bmatrix}, \qquad (2.3)$$

where $d_i \neq 0,\ i = 1, 2, \ldots, m$ and $b_i \neq 0,\ i = 1, 2, \ldots, m-1$ (we choose new notation for block entries). Since the matrix (2.3) is nonsingular then $B^+ = B^{-1}$ (see [1], for instance). This inverse can be easily found.

Proposition 1. *The entries of the matrix* $B^+ = [z_{ij}]_{m\times m}$ *are as follows: for the indices* $i = 1, 2, \ldots, m$ *we have*

$$\begin{aligned} &z_{ij} = (-1)^{i+j}\frac{1}{d_i}\prod_{s=j}^{i-1} r_s,\ j = 1, 2, \ldots, i-1; \\ &z_{ii} = \tfrac{1}{d_i}; \\ &z_{ij} = 0,\ j = i+1, i+2, \ldots, m, \end{aligned} \qquad (2.4)$$

where

$$r_s \equiv \frac{b_s}{d_s}, \quad s = 1, 2, \ldots, m-1. \qquad (2.5)$$

Based on the formulas (2.4) we can write the following simple procedure to calculate the entries of the matrix B^+.

Algorithm $(\mathbf{B} \Rightarrow \mathbf{B}^+)$**/type 1**

1. Compute the quantities r_s defined in (2.5).
2. Compute the lower triangular part of the matrix B^+. For indices $i = 1, 2, \ldots, m$:

$$z_{ii} = \frac{1}{d_i};\ z_{ij} = -r_j z_{i\,j+1},\ j = i-1, i-2, \ldots, 1.$$

End algorithm

It can be readily seen that **Algorithm** $(\mathbf{B} \Rightarrow \mathbf{B}^+)$**/type 1** requires

$$A_{ops}^{(1)} = \frac{1}{2}m^2 + O(m) \qquad (2.6)$$

arithmetical operations.

Next, we will focus our attention on computing the Moore-Penrose inverse for the blocks of type 2 and 3.

A way of computing the Moore-Penrose inverse

To solve the problem, in this section we outline an approach based upon the well-known equality

$$A^+ = \lim_{\varepsilon\to+0}(A^TA + \varepsilon I)^{-1}A^T, \qquad (3.1)$$

where I is the identity matrix, which holds true for any real matrix (see [1; 4], for instance). Here we present the main ideas to compute the Moore-Penrose inverse for a block of the type 2. For a block of the type 3, as will be seen below, the problem is reduced to the case under consideration. Note that, as in the previous case, it is convenient to introduce new notation for the entries of the block.

Let us have an $m-1 \times m$ bidiagonal matrix

$$B = \begin{bmatrix} d_1 & b_1 & & & & \\ & d_2 & b_2 & & & \\ & & \ddots & \ddots & & \\ & & & d_{m-2} & b_{m-2} & \\ & & & & d_{m-1} & b_{m-1} \end{bmatrix}, \qquad (3.2)$$

where $b_i, d_i \neq 0,\ i = 1, 2, \ldots, m-1$ and $m \geq 2$. The matrix

$$L(\varepsilon) \equiv B^T B + \varepsilon I \tag{3.3}$$

is tridiagonal matrix of the following structure:

$$L(\varepsilon) = \begin{bmatrix} d_1^2+\varepsilon & b_1 d_1 & & & \\ b_1 d_1 & d_2^2+b_1^2+\varepsilon & b_2 d_2 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & b_{m-2}d_{m-2} & d_{m-1}^2+b_{m-2}^2+\varepsilon & b_{m-1}d_{m-1} \\ & & & b_{m-1}d_{m-1} & b_{m-1}^2+\varepsilon \end{bmatrix}. \tag{3.4}$$

To invert this matrix, let us apply advantage of the algorithm developed in [5]. Consider a nonsingular symmetric tridiagonal matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & & & \\ c_{21} & c_{22} & c_{23} & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & c_{m-1\,m-2} & c_{m-1\,m-1} & c_{m-1\,m} \\ & & & c_{m\,m-1} & c_{m\,m} \end{bmatrix}. \tag{3.5}$$

We assume that $m \geq 2$. Referring to [5], the matrix $C^{-1} = [x_{ij}]_{m\times m}$ can be obtained by the following computational procedure.

Procedure 3d/inv ($C \Rightarrow C^{-1}$)

1. Compute the quantities $f_i\,(i = 2, 3, \ldots, m)$, $g_i\,(i = 2, 3, \ldots, m-1)$ and $h_i\,(i = 1, 2, \ldots, m-1)$:
$$f_i = \frac{c_{ii}}{c_{i\,i-1}},\ g_i = \frac{c_{i\,i+1}}{c_{i\,i-1}},\ h_i = \frac{c_{ii}}{c_{i\,i+1}}. \tag{3.6}$$
Note: if $m = 2$, then the quantities g_i are not introduced.
2. Compute recursively the quantities $\mu_i\,(i = 1, 2, \ldots, m)$:
$$\begin{aligned} &\mu_m = 1\,,\ \mu_{m-1} = -f_m\,, \\ &\mu_i = -f_{i+1}\mu_{i+1} - g_{i+1}\mu_{i+2}\,, \\ &i = m-2, m-3, \ldots, 1. \end{aligned} \tag{3.7}$$
3. Compute recursively the quantities $\nu_i\,(i = 1, 2, \ldots, m)$:
$$\begin{aligned} &\nu_1 = 1\,,\ \nu_2 = -h_1\,, \\ &\nu_i = -h_{i-1}\nu_{i-1} - \frac{1}{g_{i-1}}\nu_{i-2}\,,\ i = 3, 4, \ldots, m. \end{aligned} \tag{3.8}$$
4. Compute the quantity
$$t = (c_{11}\mu_1 + c_{12}\mu_2)^{-1}. \tag{3.9}$$
Note: since C is nonsingular matrix, then $c_{11}\mu_1 + c_{12}\mu_2 \neq 0$ [5].
5. The entries of the upper triangular part of the matrix C^{-1} are computed:
$$x_{ij} = \mu_j \nu_i t,\ i = 1, 2, \ldots, j\,; \quad j = 1, 2, \ldots, m. \tag{3.10}$$
6. The entries of the lower triangular part of the matrix C^{-1} are found:
$$x_{ij} = x_{ji},\ i = j+1, j+2, \ldots, m\,; \quad j = 1, 2, \ldots, m-1. \tag{3.11}$$

End procedure

The proposed way to obtain the Moore-Penrose inverse B^+ is as follows. In consistence with equality (3.1) and notation (3.3), we have

$$B^+ = \lim_{\varepsilon\to+0} L(\varepsilon)^{-1} B^T. \tag{3.12}$$

Finding first the inverse matrix $L(\varepsilon)^{-1}$, the entries of the matrix $L(\varepsilon)^{-1}B^T$ are calculated and a character of their dependence on the parameter ε is revealed. Then, according to the equality (3.12), passing to the limit when $\varepsilon \to +0$, we arrive to a closed form expressions for the entries of the matrix B^+.

The Moore-Penrose inverse of rectangular blocks

Let us consider as the matrix C from (3.5) the tridiagonal matrix $L(\varepsilon)$ obtained in (3.4). Comparing the records of these matrices, we have

$$c_{ii} = d_i^2 + b_{i-1}^2 + \varepsilon\,,\ i = 1, 2, \ldots, m \tag{4.1}$$

(in order to unify records of formulas, we set $b_0 = 0$) and

$$c_{i\,i+1} = b_i d_i\,,\ i = 1, 2, \ldots, m-1\,;$$
$$c_{i\,i-1} = b_{i-1} d_{i-1}\,,\ i = 2, 3, \ldots, m. \tag{4.2}$$

In accordance with our plan, let us carry out a more detailed analysis of the quantities successively computed in the procedure **3d/inv** from Section 3.

Consider first the quantities f_i, g_i and h_i which were introduced in (3.6). Using the expressions (4.1) and (4.2), we get

$$f_i = \mathring{f}_i + \alpha_i\varepsilon\,,\ i = 2, 3, \ldots, m,$$

where

$$\mathring{f}_i = \frac{d_i^2 + b_{i-1}^2}{b_{i-1}d_{i-1}}\,,\ \alpha_i = \frac{1}{b_{i-1}d_{i-1}}\,; \tag{4.3}$$

$$g_i = \frac{b_i d_i}{b_{i-1} d_{i-1}}\,,\ i = 2,3,\ldots,m-1; \tag{4.4}$$

$$h_i = \overset{\circ}{h}_i + \beta_i \varepsilon\,,\ i = 1,2,\ldots,m-1,$$

where

$$\overset{\circ}{h}_i = \frac{d_i^2 + b_{i-1}^2}{b_i d_i}\,,\ \beta_i = \frac{1}{b_i d_i}\,. \tag{4.5}$$

Next, go to the quantities μ_i and ν_i recursively defined in (3.7) and (3.8), respectively.

Lemma 2. *The quantities μ_i are represented as*

$$\begin{aligned} &\mu_m = \overset{\circ}{\mu}_m + \gamma_m \varepsilon\,, \quad \mu_{m-1} = \overset{\circ}{\mu}_{m-1} + \gamma_{m-1}\varepsilon\,,\\ &\mu_i = \overset{\circ}{\mu}_i + \gamma_i \varepsilon + O(\varepsilon^2)\,, \quad 1 \le i \le m-2\,, \end{aligned} \tag{4.6}$$

where the quantities $\overset{\circ}{\mu}_i$ and γ_i satisfy the following recurrence relations:

$$\begin{aligned} &\overset{\circ}{\mu}_m = 1\,,\ \overset{\circ}{\mu}_{m-1} = -\overset{\circ}{f}_m\,,\\ &\overset{\circ}{\mu}_i = -\overset{\circ}{f}_{i+1}\overset{\circ}{\mu}_{i+1} - g_{i+1}\overset{\circ}{\mu}_{i+2}\,,\\ &i = m-2, m-3, \ldots, 1 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &\gamma_m = 0\,,\ \gamma_{m-1} = -\alpha_m\,,\\ &\gamma_i = -\overset{\circ}{f}_{i+1}\,\gamma_{i+1} - g_{i+1}\gamma_{i+2} - \alpha_{i+1}\overset{\circ}{\mu}_{i+1}\,,\\ &i = m-2, m-3, \ldots, 1. \end{aligned} \tag{4.8}$$

Proof. Since $\mu_m = 1$ then in (4.6) we set $\overset{\circ}{\mu}_m = 1$, $\gamma_m = 0$. Further, $\mu_{m-1} = -f_m$ (see (3.7)). According to the expressions (4.3) we have $f_m = \overset{\circ}{f}_m + \alpha_m \varepsilon$. Therefore in the representation (4.6) we set $\overset{\circ}{\mu}_{m-1} = -\overset{\circ}{f}_m$, $\gamma_{m-1} = -\alpha_m$.

For the indices in the range $1 \le i \le m-2$, required representations can be readily derived by induction from the relations (3.7) using expressions (4.3). Indeed, having done simple transformations as follows

$$\begin{aligned} \mu_i &= -f_{i+1}\mu_{i+1} - g_{i+1}\mu_{i+2}\\ &= -(\overset{\circ}{f}_{i+1} + \alpha_{i+1}\varepsilon)(\overset{\circ}{\mu}_{i+1} + \gamma_{i+1}\varepsilon + O(\varepsilon^2))\\ &\quad -g_{i+1}(\overset{\circ}{\mu}_{i+2} + \gamma_{i+2}\varepsilon + O(\varepsilon^2))\\ &= (-\overset{\circ}{f}_{i+1}\overset{\circ}{\mu}_{i+1} - g_{i+1}\overset{\circ}{\mu}_{i+2})\\ &\quad +(-\overset{\circ}{f}_{i+1}\,\gamma_{i+1} - g_{i+1}\gamma_{i+2} - \alpha_{i+1}\overset{\circ}{\mu}_{i+1})\varepsilon\\ &\quad +O(\varepsilon^2)\,, \end{aligned}$$

we get (4.6) as well as recurrence relations (4.7) and (4.8). □

The quantities $\overset{\circ}{\mu}_i$ computed by the recursion (4.7) can be represented in closed form.

Let us introduce the following notation:

$$r_s \equiv \frac{b_s}{d_s}, \quad s = 1,2,\ldots,m-1\,. \tag{4.9}$$

Additionally, we set $r_0 = r_m = 1$.

Lemma 3. *The quantities $\overset{\circ}{\mu}_i$ can be written in the form*

$$\overset{\circ}{\mu}_i = (-1)^{m-i}\prod_{s=i}^{m-1} r_s\,, \quad i = 1,2,\ldots,m. \tag{4.10}$$

Proof. Firstly, the value $\overset{\circ}{\mu}_m = 1$ conforms to the record (4.10). Then, in accordance with (4.3) and (4.7),

$$\overset{\circ}{\mu}_{m-1} = -\overset{\circ}{f}_m = -\frac{b_{m-1}}{d_{m-1}} = -r_{m-1}\,.$$

Further reasoning is carried out by induction. Using the expressions (4.3) and (4.4), proceeding from (4.7) we obtain

$$\begin{aligned} \overset{\circ}{\mu}_i &= -\frac{d_{i+1}^2 + b_i^2}{b_i d_i}(-1)^{m-i-1}\prod_{s=i+1}^{m-1} r_s\\ &\quad -\frac{b_{i+1}d_{i+1}}{b_i d_i}(-1)^{m-i-2}\prod_{s=i+2}^{m-1} r_s\\ &= (-1)^{m-i}\prod_{s=i+2}^{m-1} r_s\left(\frac{d_{i+1}^2 + b_i^2}{b_i d_i}\, r_{i+1} - \frac{b_{i+1}d_{i+1}}{b_i d_i}\right)\\ &= (-1)^{m-i}\prod_{s=i}^{m-1} r_i\,, \end{aligned}$$

which completes the proof. □

The next assertion is a simple consequence of the formula (4.10).

Corollary 4. *The following relation holds:*

$$\overset{\circ}{\mu}_i = -r_i\,\overset{\circ}{\mu}_{i+1}\,, \quad i = 1,2,\ldots,m-1\,. \tag{4.11}$$

A representation similar to (4.6) takes place also for the quantities ν_i.

Lemma 5. *The quantities ν_i are represented as*

$$\begin{aligned} &\nu_1 = \overset{\circ}{\nu}_1 + \delta_1\varepsilon\,, \quad \nu_2 = \overset{\circ}{\nu}_2 + \delta_2\varepsilon\,,\\ &\nu_i = \overset{\circ}{\nu}_i + \delta_i\varepsilon + O(\varepsilon^2)\,, \quad 3 \le i \le m\,, \end{aligned} \tag{4.12}$$

where the quantities $\overset{\circ}{\nu}_i$ and δ_i satisfy the following recurrence relations:

$$\begin{aligned} &\overset{\circ}{\nu}_1 = 1\,,\ \overset{\circ}{\nu}_2 = -\overset{\circ}{h}_1\,,\\ &\overset{\circ}{\nu}_i = -\overset{\circ}{h}_{i-1}\overset{\circ}{\nu}_{i-1} - \frac{1}{g_{i-1}}\overset{\circ}{\nu}_{i-2}\,,\ i = 3,4,\ldots,m \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} &\delta_1 = 0\,,\ \delta_2 = -\beta_1\,,\\ &\delta_i = -\overset{\circ}{h}_{i-1}\,\delta_{i-1} - \frac{1}{g_{i-1}}\delta_{i-2} - \beta_{i-1}\overset{\circ}{\nu}_{i-1}\,,\\ &i = 3,4,\ldots,m. \end{aligned} \tag{4.14}$$

Proof. Since $\nu_1 = 1$ then in (4.12) we set $\overset{\circ}{\nu}_1 = 1$, $\delta_1 = 0$. Further, $\nu_2 = -h_1$ (see (3.8)). According to the expressions (4.5) we have $h_1 = \overset{\circ}{h}_1 + \beta_1\varepsilon$. Therefore in the representation (4.12) we set $\overset{\circ}{\nu}_2 = -\overset{\circ}{h}_1$, $\delta_2 = -\beta_1$.

For the indices in the range $3 \le i \le m$, required representations can be readily derived by induction from the relations (3.8) using expressions (4.5). Indeed, having done simple transformations as follows

$$\begin{aligned}
\nu_i &= -h_{i-1}\nu_{i-1} - \frac{1}{g_{i-1}}\nu_{i-2} \\
&= -(\overset{\circ}{h}_{i-1} + \beta_{i-1}\varepsilon)(\overset{\circ}{\nu}_{i-1} + \delta_{i-1}\varepsilon + O(\varepsilon^2)) \\
&\quad - \tfrac{1}{g_{i-1}}(\overset{\circ}{\nu}_{i-2} + \delta_{i-2}\varepsilon + O(\varepsilon^2)) \\
&= (-\overset{\circ}{h}_{i-1}\overset{\circ}{\nu}_{i-1} - \frac{1}{g_{i-1}}\overset{\circ}{\nu}_{i-2}) \\
&\quad + (-\overset{\circ}{h}_{i-1}\delta_{i-1} - \tfrac{1}{g_{i-1}}\delta_{i-2} - \beta_{i-1}\overset{\circ}{\nu}_{i-1})\varepsilon \\
&\quad + O(\varepsilon^2)\,,
\end{aligned}$$

we get (4.12) as well as recurrence relations (4.13) and (4.14). □

We can write closed form expressions for the quantities $\overset{\circ}{\nu}_i$ as well.

Lemma 6. *The quantities $\overset{\circ}{\nu}_i$ can be written in the form*

$$\overset{\circ}{\nu}_i = (-1)^{i+1}\prod_{s=1}^{i-1}\frac{1}{r_s}\,, \quad i = 1, 2, \ldots, m. \tag{4.15}$$

Proof. The value $\overset{\circ}{\nu}_1 = 1$ conforms to the record (4.15). Then in accordance with (4.5) and (4.13),

$$\overset{\circ}{\nu}_2 = -\overset{\circ}{h}_1 = -\frac{d_1}{b_1} = -\frac{1}{r_1}\,.$$

Further reasoning is carried out by induction. Taking into account the expressions (4.4), (4.5) and using (4.13) we get

$$\begin{aligned}
\overset{\circ}{\nu}_i &= -\overset{\circ}{h}_{i-1}\overset{\circ}{\nu}_{i-1} - \frac{1}{g_{i-1}}\overset{\circ}{\nu}_{i-2} \\
&= -\frac{d_{i-1}^2 + b_{i-2}^2}{b_{i-1}d_{i-1}}(-1)^i\prod_{s=1}^{i-2}\frac{1}{r_s} \\
&\quad - \tfrac{b_{i-2}d_{i-2}}{b_{i-1}d_{i-1}}(-1)^{i-1}\prod_{s=1}^{i-3}\tfrac{1}{r_s} \\
&= (-1)^{i+1}\left(\frac{d_{i-1}^2 + b_{i-2}^2}{b_{i-1}d_{i-1}}\frac{1}{r_{i-2}} - \frac{b_{i-2}d_{i-2}}{b_{i-1}d_{i-1}}\right) \\
&\quad \cdot \prod_{s=1}^{i-3}\tfrac{1}{r_s} \\
&= (-1)^{i+1}\frac{1}{r_{i-2}}\frac{1}{r_{i-1}}\prod_{s=1}^{i-3}\frac{1}{r_s} = (-1)^{i+1}\prod_{s=1}^{i-1}\frac{1}{r_i}\,,
\end{aligned}$$

which completes the proof. □

The next assertion is a simple consequence of the formula (4.15).

Corollary 7. *The following relation holds:*

$$\overset{\circ}{\nu}_{i+1} = -\frac{1}{r_i}\,\overset{\circ}{\nu}_i\,, \quad i = 1, 2, \ldots, m-1\,. \tag{4.16}$$

Our next task is to derive an expression for the quantity t given in (3.9), depending on the parameter ε. Since $c_{11} = d_1^2 + \varepsilon$, $c_{12} = b_1 d_1$ (see (4.1) and (4.2)) then taking into account the representations (4.6) for the quantities μ_i we get

$$\begin{aligned}
t &= ((d_1^2 + \varepsilon)(\overset{\circ}{\mu}_1 + \gamma_1\varepsilon + O(\varepsilon^2)) \\
&\quad + b_1 d_1(\overset{\circ}{\mu}_2 + \gamma_2\varepsilon + O(\varepsilon^2)))^{-1} \\
&= (d_1^2(\overset{\circ}{\mu}_1 + r_1\overset{\circ}{\mu}_2) \\
&\quad + (\overset{\circ}{\mu}_1 + d_1(\gamma_1 d_1 + \gamma_2 b_1))\varepsilon + O(\varepsilon^2))^{-1}.
\end{aligned}$$

By virtue of the relation (4.11), $\overset{\circ}{\mu}_1 + r_1\overset{\circ}{\mu}_2 = 0$. Thus

$$t = \frac{1}{(\overset{\circ}{\mu}_1 + d_1(\gamma_1 d_1 + \gamma_2 b_1))\varepsilon + O(\varepsilon^2)}\,. \tag{4.17}$$

Having the representations for the quantities μ_i, ν_i and t, by formulas (3.10) and (3.11) we get the entries of the inverse matrix

$$L(\varepsilon)^{-1} = [x_{ij}]_{m\times m}\,.$$

Further, let us introduce the matrix

$$Y(\varepsilon) \equiv L(\varepsilon)^{-1}B^T. \tag{4.18}$$

According to the equality (3.1) and notation (3.3), $B^+ = \lim_{\varepsilon\to+0} Y(\varepsilon)$. If

$$B^+ = [z_{ij}]_{m\times m-1}\,, \quad Y(\varepsilon) = [y_{ij}(\varepsilon)]_{m\times m-1}$$

then

$$z_{ij} = \lim_{\varepsilon\to+0} y_{ij}(\varepsilon),$$
$$i = 1, 2, \ldots, m\,,\ j = 1, 2, \ldots, m-1. \tag{4.19}$$

As follows from (4.18), the entries of the matrix $Y(\varepsilon)$ are calculated by the rule

$$y_{ij}(\varepsilon) = x_{ij}d_j + x_{i\,j+1}b_j\,. \tag{4.20}$$

Subject to the formulas (3.10) and (3.11), for a fixed index j in the range $1 \le j \le m-1$ we consider separately two cases: $i = 1, 2, \ldots, j$ and $i = j+1, j+2, \ldots, m$.

- Indices $i = 1, 2, \ldots, j$.

Taking the expression (3.10) for the entries x_{ij}, from (4.20) we can write

$$y_{ij}(\varepsilon) = t\nu_i(\mu_j d_j + \mu_{j+1}b_j). \tag{4.21}$$

Then, using the representations (4.6) of the quantities μ_i, we have

$$\begin{aligned}
\mu_j d_j + \mu_{j+1}b_j &= (\overset{\circ}{\mu}_j + \gamma_j\varepsilon + O(\varepsilon^2))d_j \\
&\quad + (\overset{\circ}{\mu}_{j+1} + \gamma_{j+1}\varepsilon + O(\varepsilon^2))b_j \\
&= (\overset{\circ}{\mu}_j d_j + \overset{\circ}{\mu}_{j+1} b_j) \\
&\quad + (\gamma_j d_j + \gamma_{j+1}b_j)\varepsilon + O(\varepsilon^2).
\end{aligned}$$

As follows from the relation (4.11),

$$\overset{\circ}{\mu}_j d_j + \overset{\circ}{\mu}_{j+1} b_j = d_j(\overset{\circ}{\mu}_j + r_j \overset{\circ}{\mu}_{j+1}) = 0.$$

Thus

$$\mu_j d_j + \mu_{j+1} b_j = (\gamma_j d_j + \gamma_{j+1} b_j)\varepsilon + O(\varepsilon^2). \quad (4.22)$$

Substituting the expression (4.22) as well as the representations (4.12) and (4.17) of the quantities ν_i and t, respectively, into the right hand side of the equality (4.21) yields

$$y_{ij}(\varepsilon) = \frac{\overset{\circ}{\nu}_i (\gamma_j d_j + \gamma_{j+1} b_j) + O(\varepsilon)}{\overset{\circ}{\mu}_1 + d_1(\gamma_1 d_1 + \gamma_2 b_1) + O(\varepsilon)}.$$

By taking limit in the previous equality, according to (4.19) we find

$$z_{ij} = \frac{\overset{\circ}{\nu}_i (\gamma_j d_j + \gamma_{j+1} b_j)}{\overset{\circ}{\mu}_1 + d_1(\gamma_1 d_1 + \gamma_2 b_1)}, \quad i = 1, 2, \ldots, j.$$

Further, let us introduce the notation

$$u_j \equiv \gamma_j d_j + \gamma_{j+1} b_j, \quad j = 1, 2, \ldots, m-1. \quad (4.23)$$

Then the entries z_{ij} can be written as follows:

$$z_{ij} = \frac{\overset{\circ}{\nu}_i u_j}{q}, \quad i = 1, 2, \ldots, j, \quad (4.24)$$

where

$$q \equiv \overset{\circ}{\mu}_1 + d_1 u_1. \quad (4.25)$$

Now let us turn to the quantities u_j defined in (4.23). For the index $j = m-1$, using the expressions (4.8) and (4.3), we have

$$u_{m-1} = \gamma_{m-1} d_{m-1} + \gamma_m b_{m-1} = -\alpha_m d_{m-1} = -\frac{1}{b_{m-1}}. \quad (4.26)$$

For the indices $j = m-2, m-3, \ldots, 1$, taking the expressions (4.8), (4.3) and (4.4), we get

$$\begin{aligned}
\gamma_j d_j &= (-\overset{\circ}{f}_{j+1} \gamma_{j+1} - g_{j+1}\gamma_{j+2} - \alpha_{j+1} \overset{\circ}{\mu}_{j+1}) d_j \\
&= -\frac{d_{j+1}^2 + b_j^2}{b_j d_j}\gamma_{j+1} d_j - \frac{b_{j+1} d_{j+1}}{b_j d_j}\gamma_{j+2} d_j \\
&\quad -\tfrac{1}{b_j d_j} \overset{\circ}{\mu}_{j+1} d_j \\
&= -\gamma_{j+1} b_j - \frac{d_{j+1}^2}{b_j}\gamma_{j+1} \\
&\quad -\tfrac{b_{j+1} d_{j+1}}{b_j}\gamma_{j+2} - \tfrac{1}{b_j} \overset{\circ}{\mu}_{j+1} \\
&= -\gamma_{j+1} b_j - \frac{d_{j+1}}{b_j}(\gamma_{j+1} d_{j+1} + \gamma_{j+2} b_{j+1}) \\
&\quad -\tfrac{1}{b_j} \overset{\circ}{\mu}_{j+1}.
\end{aligned}$$

Hence

$$\gamma_j d_j + \gamma_{j+1} b_j = -\frac{d_{j+1}}{b_j}(\gamma_{j+1} d_{j+1} + \gamma_{j+2} b_{j+1}) - \frac{1}{b_j} \overset{\circ}{\mu}_{j+1}.$$

With regard of the notation (4.23) we arrive at the equality

$$u_j = -\frac{d_{j+1}}{b_j} u_{j+1} - \frac{1}{b_j} \overset{\circ}{\mu}_{j+1}. \quad (4.27)$$

Summarizing the above considerations, on the basis of the obtained equalities (4.26) and (4.27), we can state that the quantities u_j satisfy the following relations:

$$\begin{aligned}
u_{m-1} &= -\frac{1}{b_{m-1}}, \\
u_j &= -\frac{d_{j+1} u_{j+1} + \overset{\circ}{\mu}_{j+1}}{b_j}, \\
&\qquad j = m-2, m-3, \ldots, 1.
\end{aligned} \quad (4.28)$$

The quantities u_j can be represented in closed form as well. Namely, the following statement holds.

Lemma 8. *The quantities u_j are written as*

$$u_j = \frac{(-1)^{m-j}}{d_j} \sum_{k=1}^{m-j} \left(\prod_{s=j}^{m-k} \frac{1}{r_s} \right) \left(\prod_{s=m-k+1}^{m-1} r_s \right),$$
$$j = 1, 2, \ldots, m-1. \quad (4.29)$$

The assertion can be proven by direct substituting the expression (4.29) into the relations (4.28) and using the expression (4.10) for the quantities $\overset{\circ}{\mu}_j$.

As a direct consequence of the expressions (4.10) and (4.29) we get the expression for the quantity q defined in (4.25).

Lemma 9. *The quantity q is written as*

$$q = (-1)^{m-1} \sum_{k=1}^{m} \left(\prod_{s=1}^{m-k} \frac{1}{r_s} \right) \left(\prod_{s=m-k+1}^{m-1} r_s \right). \quad (4.30)$$

Finally, let us replace the expressions (4.15), (4.29) and (4.30) of the quantities $\overset{\circ}{\nu}_i$, u_j and q, respectively, into (4.24). As a result, we obtain the following expression for the entries of the upper triangular part of the matrix B^+:

$$z_{ij} = \frac{(-1)^{i+j} \sum_{k=1}^{m-j} \left(\prod_{s=j}^{m-k} \frac{1}{r_s} \right) \left(\prod_{s=m-k+1}^{m-1} r_s \right)}{\prod_{s=1}^{i-1} r_s \cdot d_j \sum_{k=1}^{m} \left(\prod_{s=1}^{m-k} \frac{1}{r_s} \right) \left(\prod_{s=m-k+1}^{m-1} r_s \right)},$$
$$i = 1, 2, \ldots, j. \quad (4.31)$$

- Indices $i = j+1, j+2, \ldots, m$.

Using the expressions (3.10) and (3.11), from (4.20) we get the equality

$$y_{ij}(\varepsilon) = t\mu_i(\nu_j d_j + \nu_{j+1} b_j). \quad (4.32)$$

In accordance with the representations (4.12) we have

$$\begin{aligned}\nu_j d_j + \nu_{j+1} b_j &= (\overset{\circ}{\nu}_j + \delta_j \varepsilon + O(\varepsilon^2)) d_j \\ &\quad + (\overset{\circ}{\nu}_{j+1} + \delta_{j+1}\varepsilon + O(\varepsilon^2)) b_j \\ &= (\overset{\circ}{\nu}_j d_j + \overset{\circ}{\nu}_{j+1} b_j) \\ &\quad + (\delta_j d_j + \delta_{j+1} b_j)\varepsilon + O(\varepsilon^2).\end{aligned}$$

As follows from the relation (4.16),

$$\overset{\circ}{\nu}_j d_j + \overset{\circ}{\nu}_{j+1} b_j = d_j(\overset{\circ}{\nu}_j + r_j \overset{\circ}{\nu}_{j+1}) = 0.$$

Thus

$$\nu_j d_j + \nu_{j+1} b_j = (\delta_j d_j + \delta_{j+1} b_j)\varepsilon + O(\varepsilon^2). \quad (4.33)$$

Substituting the expression (4.33) as well as the representations (4.6) and (4.17) of the quantities μ_i and t, respectively, into the right hand side of the equality (4.32) yields

$$y_{ij}(\varepsilon) = \frac{\overset{\circ}{\mu}_i (\delta_j d_j + \delta_{j+1} b_j) + O(\varepsilon)}{\overset{\circ}{\mu}_1 + d_1(\gamma_1 d_1 + \gamma_2 b_1) + O(\varepsilon)}.$$

By taking limit in this equality, when $\varepsilon \to +0$, according to (4.19) we find

$$z_{ij} = \frac{\overset{\circ}{\mu}_i (\delta_j d_j + \delta_{j+1} b_j)}{\overset{\circ}{\mu}_1 + d_1(\gamma_1 d_1 + \gamma_2 b_1)}, \quad i = j+1, j+2, \ldots, m.$$

Similarly to the previous case, we introduce the notation

$$w_j \equiv \delta_j d_j + \delta_{j+1} b_j, \quad j = 1, 2, \ldots, m-1. \quad (4.34)$$

Then the entries z_{ij} can be written as follows:

$$z_{ij} = \frac{\overset{\circ}{\mu}_i w_j}{q}, \quad i = j+1, j+2, \ldots, m. \quad (4.35)$$

Consider the quantities w_j defined in (4.34). For the index $j = 1$, using the expressions (4.14) and (4.5), we have

$$w_1 = \delta_1 d_1 + \delta_2 b_1 = -\beta_1 b_1 = -\frac{1}{d_1}. \quad (4.36)$$

For the indices $j = 2, 3, \ldots, m-1$, taking the expressions (4.14), (4.4) and (4.5) yields

$$\begin{aligned}\delta_{j+1} b_j &= (-\overset{\circ}{h}_j \delta_j - \frac{1}{g_j}\delta_{j-1} - \beta_j \overset{\circ}{\nu}_j) b_j \\ &= -\frac{d_j^2 + b_{j-1}^2}{b_j d_j}\delta_j b_j - \frac{b_{j-1} d_{j-1}}{b_j d_j}\delta_{j-1} b_j \\ &\quad -\tfrac{1}{b_j d_j} \overset{\circ}{\nu}_j b_j \\ &= -\delta_j d_j - \frac{b_{j-1}^2}{d_j}\delta_j - \frac{b_{j-1} d_{j-1}}{d_j}\delta_{j-1} \\ &\quad -\tfrac{1}{d_j} \overset{\circ}{\nu}_j \\ &= -\delta_j d_j - \frac{b_{j-1}}{d_j}(\delta_j b_{j-1} + \delta_{j-1} d_{j-1}) \\ &\quad -\tfrac{1}{d_j} \overset{\circ}{\nu}_j.\end{aligned}$$

Hence

$$\delta_j d_j + \delta_{j+1} b_j = -\frac{b_{j-1}}{d_j}(\delta_{j-1} d_{j-1} + \delta_j b_{j-1}) - \frac{1}{d_j} \overset{\circ}{\nu}_j.$$

In accordance with notation (4.34) we get the equality

$$w_j = -\frac{b_{j-1}}{d_j} w_{j-1} - \frac{1}{d_j} \overset{\circ}{\nu}_j. \quad (4.37)$$

Summing up the above considerations, on the basis of the equalities (4.36) and (4.37), we infer that the quantities w_j satisfy the following relations:

$$\begin{aligned} w_1 &= -\frac{1}{d_1}, \\ w_j &= -\frac{b_{j-1} w_{j-1} + \overset{\circ}{\nu}_j}{d_j}, \quad j = 2, 3, \ldots, m-1. \end{aligned} \quad (4.38)$$

As with u_j, the quantities w_j can be represented in closed form. The following statement holds.

Lemma 10. *The quantities w_j are written as*

$$w_j = \frac{(-1)^j}{d_j} \sum_{k=1}^{j} \left(\prod_{s=1}^{k-1} \frac{1}{r_s} \right) \left(\prod_{s=k}^{j-1} r_s \right),$$

$$j = 1, 2, \ldots, m-1. \quad (4.39)$$

The assertion can be proven by direct substitution of the expression (4.39) into the relations (4.38) and by using the expression (4.15) for the quantities $\overset{\circ}{\nu}_j$.

Finally, let us replace the expressions (4.10), (4.39) and (4.30) of the quantities $\overset{\circ}{\mu}_i$, w_j and q, respectively, into the equality (4.35). Resulting formula for the entries of the lower triangular part of the matrix B^+ is of the following type:

$$z_{ij} = \frac{(-1)^{i+j+1} \left(\prod_{s=i}^{m-1} r_s \right) \sum_{k=1}^{j} \left(\prod_{s=1}^{k-1} \frac{1}{r_s} \right) \left(\prod_{s=k}^{j-1} r_s \right)}{d_j \sum_{k=1}^{m} \left(\prod_{s=1}^{m-k} \frac{1}{r_s} \right) \left(\prod_{s=m-k+1}^{m-1} r_s \right)},$$

$$i = j+1, j+2, \ldots, m. \quad (4.40)$$

Combining the above considerations, i.e. having the formulas (4.31) and (4.40), we arrive at the following statement.

Theorem 11. *Let B be an $m-1 \times m$ bidiagonal matrix given in (3.2). Then the entries of the Moore-Penrose inverse $B^{+} = [z_{ij}]_{m \times m-1}$ of this matrix are as follows:*

1) for indices $j = 1,2,\ldots,m-1$ and $i = 1,2,\ldots,j$:

$$z_{ij} = \frac{(-1)^{i+j} \sum\limits_{k=1}^{m-j} \left(\prod\limits_{s=j}^{m-k} \frac{1}{r_s} \right) \left(\prod\limits_{s=m-k+1}^{m-1} r_s \right)}{\prod\limits_{s=1}^{i-1} r_s \cdot d_j \sum\limits_{k=1}^{m} \left(\prod\limits_{s=1}^{m-k} \frac{1}{r_s} \right) \left(\prod\limits_{s=m-k+1}^{m-1} r_s \right)}; \tag{4.41}$$

2) for indices $j = 1,2,\ldots,m-1$ and $i = j+1, j+2,\ldots,m$:

$$z_{ij} = \frac{(-1)^{i+j+1} \left(\prod\limits_{s=i}^{m-1} r_s \right) \sum\limits_{k=1}^{j} \left(\prod\limits_{s=1}^{k-1} \frac{1}{r_s} \right) \left(\prod\limits_{s=k}^{j-1} r_s \right)}{d_j \sum\limits_{k=1}^{m} \left(\prod\limits_{s=1}^{m-k} \frac{1}{r_s} \right) \left(\prod\limits_{s=m-k+1}^{m-1} r_s \right)}, \tag{4.42}$$

where the quantities r_s are defined in (4.9).

Below is an example to illustrate Theorem 4.1.

Example 1. Consider $m-1 \times m$ bidiagonal matrix

$$B = \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}.$$

Calculations by the formulas (4.41) and (4.42) give the following result:

$$z_{ij} = \begin{cases} (-1)^{i+j} \left(1 - \dfrac{j}{m} \right), & i = 1,2,\ldots,j, \\ (-1)^{i+j+1} \dfrac{j}{m}, & i = j+1, j+2, \ldots, m \end{cases},$$

$$j = 1,2,\ldots,m-1.$$

Thus in Theorem 4.1 we give formulas for the entries of the Moore-Penrose inverse of a block of the type 2. In addition, based on the expressions and recurrence relations obtained in this section, we suggest a numerical algorithm to compute the entries of the matrix $B^{+} = [z_{ij}]_{m \times m-1}$.

Algorithm $(\mathbf{B} \Rightarrow \mathbf{B}^{+})$/type 2

1. Compute the quantities r_s (see (4.9)):

$$r_s = \frac{b_s}{d_s}, \; s = 1,2,\ldots,m-1; \; r_0 = r_m = 1.$$

2. Compute the quantities $\overset{\circ}{\mu}_i$ (see (4.7),(4.11)):

$$\overset{\circ}{\mu}_m = 1; \; \overset{\circ}{\mu}_i = -r_i \overset{\circ}{\mu}_{i+1}, \; i = m-1, m-2, \ldots, 1.$$

3. Compute the quantities $\overset{\circ}{\nu}_i$ (see (4.13),(4.16)):

$$\overset{\circ}{\nu}_1 = 1; \; \overset{\circ}{\nu}_{i+1} = -\frac{1}{r_i} \overset{\circ}{\nu}_i, \; i = 1,2,\ldots,m-1.$$

4. Compute the quantities u_j (see (4.28)):

$$u_{m-1} = -\frac{1}{b_{m-1}};$$

$$u_j = -\frac{d_{j+1} u_{j+1} + \overset{\circ}{\mu}_{j+1}}{b_j},$$
$$j = m-2, m-3, \ldots, 1.$$

5. Compute the quantities w_j (see (4.38)):

$$w_1 = -\frac{1}{d_1};$$

$$w_j = -\frac{b_{j-1} w_{j-1} + \overset{\circ}{\nu}_j}{d_j},$$
$$j = 2,3,\ldots,m-1.$$

6. Compute the quantity q (see (4.25)):

$$q = \overset{\circ}{\mu}_1 + d_1 u_1.$$

7. Compute the upper triangular part of the matrix B^{+} (see (4.24)):

$$z_{ij} = \frac{\overset{\circ}{\nu}_i \, u_j}{q}, \; i = 1,2,\ldots,j; \; j = 1,2,\ldots,m-1.$$

8. Compute the lower triangular part of the matrix B^{+} (see (4.35)):

$$z_{ij} = \frac{\overset{\circ}{\mu}_i \, w_j}{q},$$
$$i = j+1, j+2, \ldots, m;$$
$$j = 1,2,\ldots,m-1.$$

End algorithm

Note that the numerical implementation of the **Algorithm $(\mathbf{B} \Rightarrow \mathbf{B}^{+})$/type 2** requires

$$A_{ops}^{(2)} = m^2 + O(m) \tag{4.43}$$

arithmetical operations.

Next consider a block of type 3. Let an $m \times m-1$ bidiagonal matrix

$$B = \begin{bmatrix} d_1 & & & & \\ b_1 & d_2 & & & \\ & b_2 & \ddots & & \\ & & \ddots & d_{m-2} & \\ & & & b_{m-2} & d_{m-1} \\ & & & & b_{m-1} \end{bmatrix} \tag{4.44}$$

be given, where $b_i, d_i \neq 0$, $i = 1,2,\ldots,m-1$ and $m \geq 2$. The problem of finding the Moore-Penrose inverse of this matrix is reduced to the previous case. Indeed, by virtue of the well-known property (see [1], for instance) we have

$$B^+ = ((B^T)^+)^T, \tag{4.45}$$

where the matrix B^T is already a block of type 2 (compare (3.2) and (4.44)). Therefore as a consequence of Theorem 4.1 we can formulate the following statement.

Theorem 12. *Let B be an $m \times m-1$ bidiagonal matrix given in (4.44). Then the entries of the Moore-Penrose inverse $B^+ = [z_{ij}]_{m-1\times m}$ of this matrix are as follows:*

1) for indices $i = 1,2,\ldots,m-1$ and $j = 1,2,\ldots,i$:

$$z_{ij} = \frac{(-1)^{i+j}\sum\limits_{k=1}^{m-i}\left(\prod\limits_{s=i}^{m-k}\frac{1}{r_s}\right)\left(\prod\limits_{s=m-k+1}^{m-1} r_s\right)}{\prod\limits_{s=1}^{j-1} r_s \cdot d_i \sum\limits_{k=1}^{m}\left(\prod\limits_{s=1}^{m-k}\frac{1}{r_s}\right)\left(\prod\limits_{s=m-k+1}^{m-1} r_s\right)}; \tag{4.46}$$

2) for indices $i = 1,2,\ldots,m-1$ and $j = i+1, i+2,\ldots,m$:

$$z_{ij} = \frac{(-1)^{i+j+1}\left(\prod\limits_{s=j}^{m-1} r_s\right)\sum\limits_{k=1}^{i}\left(\prod\limits_{s=1}^{k-1}\frac{1}{r_s}\right)\left(\prod\limits_{s=k}^{i-1} r_s\right)}{d_i\sum\limits_{k=1}^{m}\left(\prod\limits_{s=1}^{m-k}\frac{1}{r_s}\right)\left(\prod\limits_{s=m-k+1}^{m-1} r_s\right)}, \tag{4.47}$$

where the quantities r_s are defined in (4.9).

Finally, with equality (4.45) in mind, we can write the following numerical algorithm to compute the entries of the matrix $B^+ = [z_{ij}]_{m-1\times m}$.

Algorithm $(\mathbf{B \Rightarrow B^+})/$type 3

1. Use **Algorithm $(\mathbf{B^T \Rightarrow (B^T)^+})/$type 2** to compute the matrix $(B^T)^+$.
2. Calculate $B^+ = ((B^T)^+)^T$.

End algorithm

As an obvious consequence of (4.43), we state that **Algorithm $(\mathbf{B \Rightarrow B^+})/$type 3** also requires

$$A^{(3)}_{ops} = m^2 + O(m) \tag{4.48}$$

arithmetical operations.

So based on the above study we can formulate the following statement.

Proposition 13. *Let a singular upper bidiagonal matrix A given in (2.1), with nonzero superdiagonal entries, be represented in the block form, according to the rule described in the Section 2 (Cases 1-4). These are blocks B_k, $k = 1,2,\ldots,p$ (see Fig. 2.5, as an illustration). Then depending on the type of a block the entries of the blocks B_k^+ in block representation of the matrix A^+ (see Fig. 2.6, as an illustration) are calculated by the formulas obtained in Proposition 2.1 ((2.4) and (2.5)), Theorem 4.1 ((4.41) and (4.42)) or Theorem 4.2 ((4.46) and (4.47)).*

To compute the Moore-Penrose inverse A^+, we have developed numerical procedures as well.

Proposition 14. *The entries of the blocks B_k^+, $k = 1,2,\ldots,p$, included in the block structure of the matrix A^+ (see Fig. 2.6) can be calculated by* **Algorithm $(\mathbf{B \Rightarrow B^+})/$type 1**, **Algorithm $(\mathbf{B \Rightarrow B^+})/$type 2** *or* **Algorithm $(\mathbf{B \Rightarrow B^+})/$type 3**, *with expenditure of arithmetical operations estimated in (2.6), (4.43) or (4.48), correspondingly.*

Below we give an example to illustrate the work of the numerical algorithms.

Example 2. Consider a matrix, which is divided into blocks as follows:

$$A = \left[\begin{array}{ccc|ccc|c|ccc}
2 & 5 & & & & & & & & \\
 & 3 & -7 & & & & & & & \\
\hline
 & & 0 & 6 & & & & & & \\
 & & & 4 & 2 & & & & & \\
 & & & & -5 & -1 & & & & \\
\hline
 & & & & & 0 & 4 & & & \\
\hline
 & & & & & & 0 & 2 & & \\
 & & & & & & & 3 & -4 & \\
 & & & & & & & & -6 & 3 \\
 & & & & & & & & & 8
\end{array}\right].$$

Applying the above algorithms, we get the following result, which coincides with the computations done with MATLAB.

$$A^+ = \left[\begin{array}{rr|rrr|r|rrrr}
0.0796 & -0.0206 & & & & & & & & \\
0.1682 & 0.0082 & & & & & & & & \\
0.0721 & -0.1393 & & & & & & & & \\
\hline
 & & 0.1667 & 0.0000 & 0.0000 & & & & & \\
 & & -0.3333 & 0.5000 & 0.0000 & & & & & \\
 & & 1.6667 & -2.5000 & -1.0000 & & & & & \\
\hline
 & & & & & 0.2500 & & & & \\
\hline
 & & & & & & 0.2006 & 0.1996 & -0.1331 & 0.0499 \\
 & & & & & & 0.0506 & -0.0337 & -0.1442 & 0.0541 \\
 & & & & & & 0.0125 & -0.0083 & 0.0055 & 0.1229
\end{array}\right].$$

Conclusion

In the work we obtained both explicit formulas and finite numerical algorithm to compute the

Moore-Penrose inverse of bidiagonal matrices. We emphasize the following important feature of the numerical algorithm. Proceeding from the structure of the blocks B_k, $k = 1, 2, \ldots, p$, in the block representation of the matrix A^+ (namely, the presence of zeros located at predetermined places) and the estimations of the number of arithmetical operations required to compute each block B_k^+ (see (2.6), (4.43) and (4.48)), we can assert that for computing one nonzero entry of the matrix A^+ asymptotically one arithmetical operation is expended. Thereby the proposed computational method can be considered as optimal. What is more, we point out another important property of the computational algorithm. The blocks B_k^+ are computed independently of each other.

References

1. A. Ben-Israel and T. N. E. Greville, *Generalized Inverses. Theory and Applications*, 2nd ed. (Springer, New York, 2003).
2. G. H. Golub and W. Kahan., "Calculating the Singular Values and Pseudo-Inverse of a Matrix". SIAM J. Num. Anal., **2**, 205–224. (1965)
3. G. H. Golub and C. Reinsch, "Singular Value Decomposition and Least Squares Solutions". Numer. Math. **14**, 403–420, (1970).
4. G. H. Golub and Ch. F. van Loan, *Matrix Computations*, 3rd ed. (The John Hopkins University Press, 1996.)
5. J. W. Lewis, "Invertion of tridiagonal matrices". Numer. Math., **38**, 333–345 (1982).

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ОБЧИСЛЕННЯ ОБЕРНЕНОГО ВІДОБРАЖЕННЯ МУРА–ПЕНРОУЗА ДЛЯ ДВУДІАГОНАЛЬНИХ МАТРИЦЬ

Обернене відображення Мура–Пенроуза є найбільш поширеним відображенням, що використовується для пошуку оберненої матриці. Це відображення має численні застосування як у теорії матриць, так і в обчислювальній лінійній алгебрі. Відомо, що обернена матриця Мура–Пенроуза може бути отримана через сингулярний розклад. Найефективніший з існуючих алгоритмів складається з двох кроків. На першому кроці, використовуючи відображення Хаусхолдера, початкова матриця зводиться до верхнього двудіагонального вигляду (алгоритм Голуба–Кахана). Другий крок відомий у науковій літературі як алгоритм Голуба–Райнша. Ця ітераційна процедура за допомогою методу Гівенса генерує послідовність двудіагональних матриць, яка збігається до діагонального вигляду. В такий спосіб отримується ітераційне наближення до сингулярного розкладу двудіагональної матриці. Головною метою цієї статті є розробка методу, який можна розглядати як альтернативну заміну алгоритму Голуба–Райнша. За допомогою реалізації запропонованого, було отримано два головні результати. По-перше, виведено явні формули для елементів обернених матриць Мура–Пенроуза для двудіагональних матриць. По-друге, використовуючи ці формули, побудовано скінченний рекурсивний алгоритм, оптимальної обчислювальної складності. Таким чином, запропоновано варіант обчислення оберненої матриці Мура–Пенроуза для двудіагональних матриць, що не використовує сингулярний розклад.

Ключові слова: псевдообернення Мура–Пенроуза, двудіагональна матриця, формула обернення, рекурсивний алгоритм.

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