DIAMETER SEARCH ALGORITHMS FOR DIRECTED CAYLEY GRAPHS

It is considered a well-known diameter search problem for finite groups. It can be formulated as follows: find the maximum possible diameter of the group over its system of generators. The diameter of a group over a specific system of generators is the diameter of the corresponding Cayley graph. In the paper a closely related problem is considered. For a specific system of generators find the diameter of corresponding Cayley graph. It is shown that the last problem is polynomially reduced to the problem of searching the minimal decomposition of elements over a system of generators.

It is proposed five algorithms to solve the diameter search problem: simple down search algorithm, fast down search algorithm, middle down search algorithms, homogeneous down search algorithm and homogeneous middle down search algorithm.

The first two algorithms are universal. They can be applied to any finite group and its systems of generators. Moreover, the fast down search algorithm is an optimized version of the simple down search algorithm.

A property of strict growing for a system of generators is introduced. In this case the search process can be optimized by focusing only on those group elements, for which minimum decompositions potentially have the maximum possible length. Based on this property middle down search algorithm is introduced.

The main part of the paper is homogeneous theory. It is considered a series of groups with its systems of generators and some additional properties of them. It is defined a homogeneous property of these series. A binary equivalence relation relies on it. The main purpose of defining such a relation is preserving decompositions of elements from the same equivalence class. It is enough to find the minimum decomposition of only one representative of the equivalence class.

It is introduced homogeneous down search and homogeneous middle down search algorithms. These algorithms can be applied to groups that belong to homogeneous series of groups with systems of generators.

For every algorithm its correctness is shown. The complexity estimations for algorithms is discussed.

**Keywords:** Cayley graph, diameter of group, system of generators.

**Introduction**

The diameter search problem in group theory can be formulated as follows: for a finite group $G$ find the maximum of its diameters $D(G)$ over all systems of generators of $G$. A few general results in this area are known. The most general conjecture was proposed by L. Babai and A. Seress in [1, Conjecture 1.7]:

**Conjecture 1.** If $G$ is a non-abelian finite simple group of order $N$, then $D(G) < (\log N)^C$ for the absolute constant $C$.

The first family of finite simple groups, for which this conjecture was proved by H. Helfgott in [2], is $PSL_2(\mathbb{Z}/p\mathbb{Z})$, where $p$ is a prime. For groups of Lie type an upper bound of the diameter was found by E. Breuillard, B. Green, and T. Tao in [3] and by L. Pyber and E. Szabó in [4]. For permutation groups upper bounds of diameters were presented by H. Helfgott and A. Seress in [5].

We can also consider in this domain another widely known problem, the minimum-length generators sequence search problem. Specifically, for a given finite group $G$, its system of generators $S$ and a target element $g \in G$ find a shortest generator sequence realizing $g$. In particular, for permutation groups such a problem is NP-hard [6].

As a partial case of the diameter search problem one can deal with the diameter search for a finite group over a fixed system of generators. For example the diameter of $Sym(n)$ over $S = \{(1, k)| k \in \{1, 2, \ldots, n\}$ was found in [7].

In this paper we consider the diameter search problem for directed Cayley graphs. We introduce different algorithms and discuss their properties.

The paper is organized as follows:

1. Section Introduction contains basic notations and decomposition problem description. The relation between decomposition problem and diameter search problem is demonstrated.

2. Section Simple down search algorithm introduces universal diameter search algorithm and
proves its correctness.
3. Section Fast down search algorithm introduces
infinite decomposition trees and optimized version
of simple down search algorithm. It is proved cor-
rectness of the algorithm.
4. Section Middle down search algorithm intro-
duces strict growing property of a system of gener-
ators and diameter search algorithm, based on it.
It is proved correctness of this algorithm.
5. Section Homogeneous theory introduces
groups-generators series, properties of them and
homogeneous equivalence between elements of
groups.
6. Section Homogeneous down search algorithm
introduces algorithm, which requires homogeneous
property of groups-generators series. It is proved
its correctness.
7. Section Homogeneous middle down search al-
gorithm introduces diameter search algorithm, which
requires both homogeneous property of groups-
generators series and strict growing of system of
generators. It is proved correctness of the algo-

**Preliminaries**

Unless otherwise specified in the paper we de-
ote by $G$ a finite group and by $S$ a system of
generators of $G$.

**Basic notation.** The following notations will
be used:
1. $k \mod m$ — the remainder of division of integer
   $k$ by integer $m \neq 0$.
2. $\overline{m_1, m_2}$ — the set of natural numbers
   $\{n_1, \ldots, n_2\}$, where $n_1 \leq n_2$.
3. $f \circ g := g(f)$ — the right composition of map-
   pings $f, g$.

We define an index tuple $I$ as a tuple of pair-
wise different natural numbers, i.e. for some $n \geq 0$
we have

$$I = (i_1, i_2, \ldots, i_n), \quad i_j \neq i_k, j \neq k.$$ 

In other words, every index tuple is a linearly or-
dered finite set of natural numbers. We call $n$ the
-cardinality of the index tuple $I$. Sometimes we
abuse terminology and refer to index tuples as to
sets with no ordering.

Let $I, J$ be disjoint index tuples (i.e. they have
no common elements) with cardinalities $n_1, n_2$ re-
spectively. Then we define the concatenation of
them as

$$I \sqcup J = (i_1, \ldots, i_{n_1}, j_1, \ldots, j_{n_2}).$$

Let $I, J$ be index tuple with cardinalities $n_1, n_2$
respectively. Then their difference is defined as

$I \setminus J$ — the tuple of numbers from the set $I \setminus J,$
ordered as in $I$.

Note that $I \setminus J$ can be empty.

**Diameter search problem.**

**Definition 1.** The (right) Cayley graph of $G$
over $S$ is a colored directed graph $\text{Cay}(G, S)$
constructed as follows:
1. the set of vertices is $G$;
2. the set of colors is $S$;
3. for any $g \in G$ and $s \in S$, the vertices $g$ and $g \cdot s$
   are connected by a directed edge of color $s$.

Since $S$ generates $G$ the Cayley graph of $G$ over
$S$ is a strongly connected graph.

Remind that the distance between two vertices
in a directed strongly connected graph is the length
of the shortest oriented path which connects them.
The diameter of the graph is the maximum of dis-
tances between its vertices.

**Definition 2.** The diameter of the group $G$ with
respect to the system of generators $S$ is the di-
амeter of the corresponding Cayley graph $\text{Cay}(G, S)$
of the group $G$ over $S$:

$$D_S(G) = D(\text{Cay}(G, S)).$$

**Definition 3.** The diameter of the group $G$ is de-
efined as the maximum of diameters of $G$ over its
systems of generators:

$$D(G) = \max_{S \in G} D_S(G).$$

**Decomposition problem.** Every element $g$ of
$G$ can be decomposed into a product

$$g = \prod_{k=1}^{l} s_{k}$$
of generators from $S$ for some natural $l$. Corre-
sponding tuple of generators $(s_1, \ldots, s_l)$ will be
called a decomposition of the element $g$ over $S$.
The length $|g|_S$ of the element $g$ over $S$ is the
length of the shortest decomposition of $g$ over $S$.

Let us formulate the following computational
problem.

**Decomposition problem:** for a given group $G$
and its system of generators $S$ find the maximum
of lengths of its elements over $S$.

Let $g_1, g_2$ be vertices from $\text{Cay}(G, S)$. Denote
by $d(g_1, g_2)$ the distance between $g_1$ and $g_2$
over $S$.

**Theorem 1.** The diameter search problem is
polynomial-time reducible to the decomposition
problem.

**Proof.** Let $l$ be the diameter of the group $G$
with respect to the system of generators $S$. It means
that there exist vertices $g_1, g_2$ from $\text{Cay}(G, S)$
such that $d(g_1, g_2) = l$. It immediately implies
$d(g_1, g_2) = d(e, h)$, where $h = g_2 \cdot g_1^{-1}$. Hence,
the labels of the shortest path between $e$ and $g_2 \cdot g_1^{-1}$
form a decomposition of $h$ over $S$. The statement
immediately follows.
An element \( a \in G \) will be called diameter element of group \( G \) over \( S \) if its length over \( S \) equals to diameter of group \( G \) over \( S \):

\[
|a|_S = D_S(G).
\]

**Simple down search algorithm**

Consider an algorithm of finding diameters that are based on breadth-first search algorithm [8] for graphs.

**Algorithm 1:** Simple down search algorithm

**Input:** \( G \) — a group, \( S \) — its system of generators  
**Result:** Diameter \( D_S(G) \) 

**Initialization:** \( \text{found} = \{e\} \), \( \text{all} = \{g | g \in G\} \), \( \text{current\_level} = \{e\} \), \( \text{level} = 0 \) 

**while** \( \text{found} \neq \text{all} \) **do** 

\[
\begin{align*}
\text{current\_level} &= \text{current\_level} \cdot S; \\
\text{found} &= \text{found} \cup \text{current\_level}; \\
\text{level} &= \text{level} + 1;
\end{align*}
\]

**end**

**Output:** level

**Theorem 2.** Simple down search algorithm is correct.  

**Proof.** We need to show that:  
1. the algorithm has no “dead” loops;  
2. the output of the algorithm is the diameter \( D_S(G) \).

These two parts will be proved separately. 

**Part 1.** Let \( a \) be arbitrary element of the group \( G \). Since \( S \) generates \( G \) there exists a decomposition of \( a \) over \( S \):

\[
a = \prod_{k=1}^{l} s_{i_k}.
\]

Therefore, the element \( a \) will belong to \( \text{found} \) at the moment when \( \text{level} = l \).

Since the group \( G \) is finite there exists \( n \) such that at the moment \( \text{level} = n \) we obtain \( \text{found} = \text{all} \).

**Part 2.** Let us denote \( D_S(G) \) by \( d \). Suppose that \( d \neq \text{level} \), where \( \text{level} \) is the output of simple down search algorithm for group \( G \) over \( S \). Consider two cases. 

1. Assume that \( d < \text{level} \). Then, for every element \( a \) of the group \( G \) there exists its decomposition over \( S \) with length \( l \leq d \). The set \( \text{found} \) is redefined in the algorithm on each loop. Hence, at the moment \( \text{level} = d \) we have:

\[
\text{all} = \text{found} = \{e\} \cup \bigcup_{l=1}^{d} S \ldots S,
\]

which means that \( \text{level} \leq d \). A contradiction.

2. Assume that \( d > \text{level} \). Then there exists an element \( a \) of the group \( G \) with length \( d \) over \( S \). By the definition of length over system of generators there are no sets of indices \( \{i_1, i_2, \ldots, i_l\}, l < d \) such that:

\[
a = \prod_{k=1}^{l} s_{i_k}.
\]

Therefore, the set \( \text{found} \) does not contain the element \( a \) when the algorithm stops. This leads to a contradiction with the requirement that \( \text{found} \) equals to \( \text{all} \).

The proof is complete.

**Proposition 3.** Let \( G \) be a finite group generated by \( S \), \( |S| = n \) and \( D_S(G) = m \). Then the total number of multiplications in simple down search algorithms is bounded from above by \( \frac{n^{m-1}}{m} \).

**Proof.** At the moment \( \text{level} = k + 1 \) the algorithm needs to multiply every element of the previous level by every generator. Then we obtain the following number of multiplications: \( |\text{current\_level}| \cdot |S| = |S|^k \cdot |S| = n^{k+1} \). As the result, the total number of multiplications will be:

\[
\sum_{k=1}^{m} n^{k+1} = \frac{n \cdot (n^m - 1)}{n - 1}.
\]

The proof is complete.

**Fast down search algorithm**

We need to define additional structures in order to describe another algorithms, in particular fast down search algorithm. After that we will prove a few statements to connect simple down search algorithm and fast down search algorithm.

**Finite decomposition tree.** Let \( G \) be a finite group, \( S = \{s_1, s_2, \ldots, s_n\} \) be its system of generators. Consider infinite rooted \( m \)-ary tree \( T(V, E) \). We introduce enumeration of vertices on each level of this tree. The vertices of the \( l \)-th level will be enumerated by numbers from 1 to \( m^l \), \( l \geq 0 \). We obtain that:

1. the root is the first vertex of level 0.  
2. the \( k \)-th child of the \( l \)-th vertex of level \( l \) will have index \((l - 1) \cdot m + k\) on level \((l + 1)\).

We also label vertices and edges of the tree \( T(V, E) \) starting from level 0 as follows:

1. the root will be labeled by \( e \).  
2. the edge, which connects the \( k \)-th vertex of level \( l \) with \([(k/m) + 1] \)th vertex of level \((l - 1)\), will be labeled by \( k \bmod m, k \in 1, \ldots, m^l \).

3. the \( k \)-th vertex of level \( l \) will be labeled by the result of product: \( b \cdot s_{k \bmod m} \), where \( b \) is the label of \([(k/m) + 1] \)th vertex of level \((l - 1), k \in 1, \ldots, m^l \).
We call such a tree the infinite decomposition tree of the group $G$ over the system of generators $S$. A path in this tree will be identified with the sequence of labels on edges along this path.

Let $T$ be the infinite decomposition tree of the group $G$ over $S$.

**Lemma 4.** An element $a$ from $G$ has decomposition $\prod_{k=1}^{l} s_{i_k}$ if and only if the path $i_1, i_2, \cdots, i_l$ connects the root vertex with the vertex labeled by $a$ in $T$.

**Proof.** Induction on the decomposition length $l$.

**Necessity.** Let $a = s_{i_k}$ for $k \in 1, \ldots, m$. The equality $e \cdot s_{i_k} = a$ implies that the $k$th vertex on level 1 will have the label $a$.

**Sufficiency.** Let the $k$th vertex of the first level be labeled by $a$. Then, from the definition of the infinite decomposition tree we have $a = e \cdot s_{i_k}$. Then $a = s_k$. Hence, $a$ has a decomposition of length $1$, i.e., $s_{i_k}$.

**Induction step:** case $l+1$ under assumption that for $l$ the statement holds.

**Necessity.** Let $a = \prod_{k=1}^{l+1} s_{i_k}$. Under inductive assumption for the element $b = \prod_{k=1}^{l} s_{i_k}$ we have: the path $i_1, i_2, \ldots, i_l$ connects the root with the vertex $w$ labeled by $b$. The equality $a = b \cdot s_{i_{l+1}}$ implies that the $(i_{l+1})$th child of the vertex $v$ of the vertex $w$ is labeled by $a$. Hence, $i_1, i_2, \ldots, i_{l+1}$ is a path from the root to $v$.

**Sufficiency.** Let $i_1, i_2, \ldots, i_{l+1}$ be a path, which connects the root with the vertex $v$ labeled by $a$. The definition of the infinite decomposition tree implies the equality $a = b \cdot s_{i_{l+1}}$, where $b$ is a label of the vertex $w$, the parent of the vertex $v$.

The product $\prod_{k=1}^{l} s_{i_k}$ equals the element $b$. Therefore, the product $\prod_{k=1}^{l+1} s_{i_k} \cdot s_{i_{l+1}} = s_{i_k}$ equals to the element $a$.

**Proposition 5.** Let $G$ be a group, $S$ be its system of generators, $l$ be a natural number.

1. The diameter of the group $G$ over the system of generators $S$ equals to $l$ if and only if $l$ is the smallest level number in $T$ such that every element of $G$ appears at least once as a label of a vertex starting from level 0 up to level $l$.

2. In simple down search algorithm an element $a \in G$ appears at the moment level $= l$ if and only if there exists a path $i_1, i_2, \ldots, i_l$ which connects the root with the vertex $v$ labeled by $a$ in $T$.

**Proof.** 1. The diameter of the group $G$ over $S$ equals to $l$ if and only if for every element $a$ of $G$ there exists a decomposition over $S$ with length $\leq l$. The last statement holds if and only if there exists a path with length $\leq l$ which connects the root with a vertex labeled by $a$. Therefore, for every element $a$ of $G$ there exists at least one vertex labeled by $a$ on levels from 0 to $l$.

2. The element $a$ appears in the simple down search algorithm at the moment level $= l$ if and only if there exists a sequence of generators $s_{i_1}, s_{i_2}, \ldots, s_{i_l} \in S$ such that $a = \prod_{k=1}^{l} s_{i_k}$. From Lemma 4 it follows that the last statement holds if and only if the path $i_1, i_2, \ldots, i_l$ connects the root with vertex labeled by $a$.

The proof is complete.

Let $v$ be a vertex of the tree $T$ on level $l$. Recall that the sub-tree $T|_v$ of $T$ rooted at the vertex $v$ is the tree constructed from $T$ as follows:

1. the root of new tree $T|_v$ is $v$.
2. the $k$th level of tree $T|_v$ consists of vertices from $(l+k)$th level of $T$ which are directly connected to $(l+k-1)$th level of $T|_v$, $l \geq 1$. Labels of edges and vertices are preserved.

Denote by $g_v$ the label of a vertex $v$ in $T$.

**Lemma 6.** Let $v, w$ be vertices of $T$ such that the labels of $v$ and $w$ are equal. Then the rooted trees $T_v$ and $T_w$ are isomorphic as labelled graphs.

**Proof.** Note, that $T|_v$ and $T|_w$ are isomorphic as rooted $m$-ary trees. The natural isomorphism $\tau$ preserving enumeration of vertices on levels is defined as follows:

1. the $k$th vertex of the $l$th level of $T|_v$ is mapped to the $k$th vertex of the $l$th level of $T|_w$, $k \in I, |C|$, $l \geq 0$;
2. an edge, which connects two vertices of the tree $T|_v$, is mapped to the edge, which connects images of corresponding vertices.

It is enough to show that isomorphism $\tau$ preserves labels of vertices.

Let $a$ and $b$ be labels of $j$th vertices on level $l$ of corresponding trees $T|_v$ and $T|_w$. Suppose that $a \neq b$. Then

$$g_v \cdot \prod_{k=1}^{l} s_{i_k} \neq g_w \cdot \prod_{k=1}^{l} s_{i_k}.$$ 

Hence, $g_v \neq g_w$. This leads to a contradiction with the equality of labels of $v$ and $w$.

The proof is complete.

Denote by $Path_T(v, w)$ the shortest path from vertex $v$ to vertex $w$ in $T$. 
Lemma 7. Let an element \( a \in G \) decomposes as 
\[
a = \prod_{k=1}^{t} s_{i_k}
\]
in set \( S \). If there exists \( t \in \prod \) such that 
the element \( \prod_{k=1}^{t} s_{i_k} \) appears as a label of a vertex 
of \( T \) on level less then \( t \) then \( |a|_S < l \).

Proof. Denote by \( b \) the product \( \prod_{k=1}^{t} s_{i_k} \). Then 
the vertex \( v \in V \) which is defined by the path 
i_1, i_2, \ldots, i_t \) starting from the root, will have label \( b \). Note, that the element \( a \) decomposes as a 
product \( \prod_{k=1}^{t} s_{i_k} \) if and only if the path \( i_1, i_2, \ldots, i_t \) 
connects the root of \( T_v \) with the vertex labeled by 
g_v \cdot a.

The assumption of the lemma implies that 
there exists a vertex \( w \in V \) such that \( w \) is upper 
than \( v \) in \( T \) and \( w \) is also labeled by \( b \). Since 
labelled trees \( T_v \) and \( T_w \) are isomorphic the vertices, 
which are defined by the path \( i_1, i_2, \ldots, i_t \) 
starting from the root in trees \( T_v \) and \( T_w \), have the same 
label \( a \). From Lemma 4 it follows that 
\[
a = \prod_{k=1}^{t} s_{i_k} = \prod_{k \in \text{Path} \_T(e,v)} s_k \cdot \prod_{k=t+1}^{t} s_{i_k} =
\prod_{k \in \text{Path} \_T(e,w)} s_k \cdot \prod_{k=t+1}^{t} s_{i_k}.
\]

Since \( w \) is upper than \( v \), the length of the path 
\( \text{Path} \_T(e,w) \) is less than \( t \). This leads to the inequality 
\( |a|_S < l \).

The proof is complete.

Fast down search algorithm. In order to 
optimise the simple down search algorithm we use 
the results of the previous section. The main goal 
is to reduce the number of multiplications.

Algorithm 2: Fast down search algorithm

Input: \( G \) — a group, \( S \) — its system of generators

Result: Diameter \( D_S(G) \)

Initialization: \( \text{found} = \{e\} \), 
\( \text{all} = \{g|g \in G\} \), \( \text{current\_level} = \{e\} \), 
\( \text{level} = 0 \);

while \( \text{found} \neq \text{all} \) do

\( \text{current\_level} = \)
\( = (\text{current\_level} \cdot S)\) \( \text{found} \);

\( \text{found} = \text{found} \cup \text{current\_level} \);

\( \text{level} = \text{level} + 1 \);

end

Output: \text{level}

Theorem 8. Fast down search algorithm is correct.

Proof. We need to show that:

1. the algorithm has no dead loops;
2. the output of the algorithm is the diameter \( D_S(G) \).

These two parts will be proved separately.

Part 1. There are no dead loops if and only if 
there exists a natural number \( n \) such that the algorithm 
will find all elements of group \( G \) (set \( \text{found} \)) 
at the moment \( \text{level} = n \).

Suppose that there exists an element \( a \in G \) 
which never appears in the set \( \text{found} \). Consider a 
decomposition \( \prod_{k=1}^{t} s_{i_k} \) of \( a \) over \( S \). Since \( a \) is not 
contained in \( \text{found} \), there exists \( t \in \prod \) such that 
the element \( b = \prod_{k=1}^{t} s_{i_k} \) appeared on an earlier 
iteration of the algorithm. This means that there 
exists a shorter decomposition of \( b \) over \( S \). From 
Lemma 4 it follows that the element \( b \) is a label of 
a vertex on the level which is upper then level \( t \).

Lemma 7 implies the inequality \( |a|_S < l \). Hence, 
for every decomposition of \( a \) a shorter decompositon 
can be found. This immediately leads to a 
contradiction for the set of all lengths of decompositions 
of \( a \) over \( S \) is bounded from below.

Part 2. Let \( d_1 \) be the output of the simple 
down search algorithm with input \( G \) and \( S \) and let 
\( d_2 \) be the output of the fast down search algorithm. 
Note that directly from these definitions we have the inequality 
\( d_1 \leq d_2 \).

Suppose, that \( d_1 < d_2 \). Then there exists an element 
of the group \( G \) such that it firstly appeared strongly after 
the \( d_1 \)th step of the fast down search algorithm. 
Otherwise, first down search algorithm stops 
at the moment \( d_1 \).

Let the element \( a \in G \) be such that:

1. \( a \) firstly appeared at the moment \( \text{level} = d_2 \) 
in the fast down search algorithm; 
2. \( a \) firstly appeared at the moment \( \text{level} = d_2 - r \) 
in the simple down search algorithm.

The second condition leads to the equality \( |a|_S = 
= d_2 - r \). Proposition 5 implies that there exists a 
path \( i_1, i_2, \ldots, i_{d_2 - r} \), which connects the root with 
the vertex labeled by \( a \). Based on the fast diameter 
search algorithm, there exists a natural number \( t \), 
\( t \leq d_2 - r \), such that the element \( b = \prod_{k=1}^{t} s_{i_k} \) 
appears earlier then \( \text{level} = (d_2 - r) \). Lemma 7 
implies that \( |a|_S < d_2 - r \). A contradiction.

The proof is complete.

The main optimization of the fast down search 
algorithm compared to the simple down search 
algorithm is to skip previously founded elements 
of a group. The number of repetitions of elements 
depends on a group and its system of generators. 
Therefore, in general the number of multiplication,
which are required for the fast down search algorithm, can be estimated only as in Proposition 3 by $\frac{1}{n-1}$. However, in some cases this number can be reduced significantly.

**Middle down search algorithm**

In this section we present an algorithm that requires additional properties of generators.

**Strictly growing system of generators.**

An element $a \in G$ will be called *properly generated* over $S$ if element for arbitrary $A \subset S, A \neq S$ we have $a \not\in (A)$.

This definition immediately leads to the following statements.

**Lemma 9.** *Every decomposition of a properly generated element contains every generator of $S$.***

**Proof.** Since every element belongs to the subgroup generated by the elements that appeared in its decomposition the statement follows.

**Lemma 10.** *The minimum possible length of a properly generated element over $S$ is $|S|$.***

**Proof.** Immediately follows from Lemma 9.

A system of generators $S$ of a group $G$ will be called *strictly growing* if every diameter element $a$ from $G$ is properly generated.

**Lemma 11.** *Let $G$ be a finite group, $S$ be its strictly growing system of generators. Then the diameter of the group $G$ over $S$ is greater or equal to $|S|$.***

**Proof.** By Lemma 10 every diameter element has length over $S$ greater or equal to $|S|$. This means that the diameter of $G$ over $S$ is not less than $|S|$.

The proof is complete.

**Middle down search algorithm.** Let $G$ be a finite group, $S$ be its strictly growing system of generators.

We introduce the following notions:

1. $G_f$ — the set of all properly generated elements of the group $G$;
2. $D_f(S,m)$ — the set of all decompositions over $S$ with length $m$ such that every generator of $S$ appears at least once in every decomposition, $m \geq |S|$;
3. $P$ — the function on the set of decompositions, which converts a decomposition to the corresponding element.

Let $a$ be an element of $G$ with a decomposition in $D_f(S,m)$. Note, that in general it does not imply that $a$ is properly generated.

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<td><strong>Result:</strong> Diameter $D_S(G)$</td>
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<tr>
<td><strong>Initialization:</strong> $found = \emptyset, all = G_f, level =</td>
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<tr>
<td><strong>while</strong> $found \neq all$ do</td>
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<td>1. $level = level + 1$;</td>
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<td>2. for $decomp \in D_f(S, level)$ do</td>
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<td><strong>Output:</strong> level</td>
</tr>
</tbody>
</table>

**Theorem 12.** *Middle down search algorithm is correct.*

**Proof.** Since the system of generators $S$ is strictly growing every diameter element is properly generated. Then the set of all diameter elements is a subset of $G_f$. This means that the diameter can be found as the lengths over $S$ of elements from $G_f$ are found. More precisely, the diameter is the maximum of these lengths:

$$D_S(G) = \max_{el \in G_f} |el|_S.$$

Hence, the main loop of the algorithm terminates after finite number of steps, i.e. after $D_S(G) - |S| + 1$ steps.

Lemma 9 implies that for every properly generated element a minimum decomposition belongs to $D_f(S, level)$ for some natural number level. From Lemma 10 it follows that level is not less than $|S|$. This explains why the main loop starts from $level = |S|$.

The proof is complete.

**Proposition 13.** *Let $G$ be a finite group generated by a strictly growing system of generators $S$, $|S| = n$ and $D_S(G) = m$ for some $n, m \in \mathbb{N}$. Then the number of multiplications required by the middle down search algorithm is bounded from above by***

$$\sum_{t=n}^{m} (t-1) \cdot \sum_{k=0}^{n-1} (-1)^k \cdot \binom{n}{k} \cdot (n-k)^t.$$

**Proof.** Since the system of generators $S$ is strictly growing the inequality $m \geq n$ holds. Hence, we need to obtain a product of every sequence of generators of length from $n$ to $m$. Moreover, every such sequence must contain every generator from $S$.
at least once. The total number of decompositions on the \( n \)th step equals
\[
\sum_{k=0}^{n-1} (-1)^k \cdot \binom{n}{k} \cdot (n-k)^t.
\]
Therefore, on the \( n \)th iteration of the algorithm the number of multiplications is not greater than
\[
(t-1) \cdot \sum_{k=0}^{n-1} (-1)^k \cdot \binom{n}{k} \cdot (n-k)^t.
\]
Hence, the total number of all multiplications from the \( n \)th to the \( n \)th step can be estimated from above as
\[
\sum_{t=1}^{m} (t-1) \cdot \sum_{k=0}^{n-1} (-1)^k \cdot \binom{n}{k} \cdot (n-k)^t.
\]
The proof is complete.

**Homogeneous theory**

In this section we consider series of groups with its systems of generators. We put additional conditions on them and obtain some useful properties. Then it gives us a possibility to introduce new algorithms.

**Inductive limits of groups.** Recall the notion of inductive limit of groups. Let \( (I, \prec) \) be a directed set, \( \{G(i) | i \in I\} \) be a family of indexed groups. Assume that there exist homomorphisms \( h_{i,j} : G(i) \rightarrow G(j), i, j \in I, i < j, \) such that
1. \( h_{i,i} = id \) over \( G(i) \) for every \( i \in I; \)
2. \( h_{i,k} = h_{j,k} \circ h_{i,j} \) for every \( i, j, k \in I, i < j < k. \)

For indices \( i, j \in I \) and elements \( x \in G(i), y \in G(j) \) we write \( x \sim y \) if there exists \( k \in I \) such that
\[
h_{i,k}(x) = h_{j,k}(y).
\]
Then \( \sim \) is an equivalence relation on the disjoint union of given groups that admits to define multiplication of equivalence classes induced by multiplication rules in given groups.

**Definition 4.** The inductive limit of the system \( (G(i), h_{i,j}) \), \( i, j \in I \) is the group defined as
\[
\lim_{\rightarrow} G(i) = \bigsqcup_{i \in I} G(i)/ \sim.
\]

**Homogeneous system of generators.** Let \( G(1) < G(2) < \ldots < G(n) < \ldots, n \in \mathbb{N} \) be an ascending group series. Let \( i, j \) be natural numbers, \( i < j \). We define the homomorphism \( h_{i,j} \) from \( G(i) \) to \( G(j) \) as the embedding mapping between these groups, i.e.
\[
h_{i,j}(g) = g, g \in G(i).
\]
Then the inductive limit of the system \( (G(i), h_{i,j}), i, j \in \mathbb{N} \) is well-defined.

**Definition 5.** A groups-generators series \( G \) is the sequence of pairs \( (G(n), SoG(n)) | n \in \mathbb{N} \) such that:
1. \( G(1) < G(2) < \ldots < G(n) < \ldots \) is an ascending group series;
2. \( SoG(n) \) is a system of generators of \( G(n) \) and
\[
SoG(n) \subset SoG(n+1), n \in \mathbb{N}.
\]

Let \( G \) be a groups-generators series. Denote by \( IL(G) \) the inductive limit of the system \( (G(i), h_{i,j}) \), \( i, j \in \mathbb{N} \) with embedding mappings \( h_{i,j}. \)

Denote by \( GDif f(n) \) the set of generators, which appear exactly on the \( n \)th, \( n \geq 1 \), i.e.
1. \( GDif f(1) = SoG(1); \)
2. \( GDif f(n) = SoG(n) \backslash SoG(n-1), n \geq 2 \).

**Definition 6.** The groups-generators series \( G \) is called uniform if:
\[
\bigcup_{k=1}^{t} GDif f(i_k) \simeq G(t),
\]
for every index tuple \( I = (i_1, i_2, \cdots, i_t) \) of cardinality \( t. \)

Let \( C \) be a natural number.

**Definition 7.** The groups-generators series \( G \) is called \( C \)-stable if:
\[
|GDif f(t)| = C, t \geq 1.
\]

Let the groups-generators series \( G \) be \( C \)-stable. Suppose that elements from \( \bigcup_{n \geq 1} SoG(n) \) are enumerated
\[
\bigcup_{n \geq 1} SoG(n) = \{s_i | s_i \in G | i \in \mathbb{N}\}
\]
and the following conditions hold:
1. \( SoG(n) = \{s_{1}, s_{2}, \ldots, s_{C}, s_{C+1}, \ldots, s_{n.C}\}, n \geq 1 \)
2. \( GDif f(n) = \{s_{(n-1).C+1}, s_{(n-1).C+2}, \ldots, s_{n.C}\}, n \geq 1 \).

Let \( I = (i_1, i_2, \ldots, i_t) \) be an index tuple. Define the mapping \( h_{C}^{I} \) from \( \mathbb{N} \) to \( t \cdot C \) by the rule:
\[
h_{C}^{I}(x) = (i_{(x-1)/C+1} - 1) \cdot C + (x-1) mod C + 1
\]
and note that the unique representation of \( x = (k-1) \cdot C + r, k \in \mathbb{N}, r \in \mathbb{C}, \) leads to the equality
\[
h_{C}^{I}(x) = (i_{k} - 1) \cdot C + r. \tag{1}
\]
Note that the last equality can be reinterpreted as follows: if \( x \) is the index of the \( t \)th generator of \( GDif f(k), \)
then \( h_G^i(x) \) is the index of the \( r \)th generator of \( GDiff(i_k) \).

Now define a mapping 
\[
\psi_G^i : SoG(n) \rightarrow \bigcup_{k=1}^n GDiff(i_k)
\]
by the rule:
\[
\psi_G^i(s_i) = s_{h_G^i(i)}.
\]

We will use notations \( \psi_G^i, h_G^i \).

**Definition 8.** A uniform and \( C \)-stable groups-generators series \( G \) is called homogeneous if for every natural \( t \) and every index tuple \( I \) of cardinality \( t \) the mapping \( \psi_G^i \) can be extended to the group isomorphism between \( G(t) \) and \( G(t) \).

We will omit the letter \( C \) in notations \( \psi_G^i, h_G^i \). We will use notations \( \psi, h \) instead, unless otherwise stated in this paper.

**Homogeneous equivalence.** Let \( G \) be a homogeneous groups-generators series. We define a binary relation \( \equiv \) on \( IL(\mathbb{G}) \).

**Definition 9.** Let \( a, b \) be elements from \( IL(\mathbb{G}) \).

We write \( a \equiv b \) if there exist index tuples \( I, J \) of the same cardinality \( n \) such that:
1. \( a \in G_I(n) \);
2. \( b \in G_J(n) \);
3. \((\psi_I^{-1} \circ \psi_J)(a) = b \).

**Lemma 14.** The binary relation \( \equiv \) is an equivalence.

**Proof.** Reflexivity. Let \( a \) be an element from \( IL(\mathbb{G}) \). The definition of the inductive limit implies the existence of natural \( n \) such that \( a \in G(n) \).

Then for the index tuple \( I = (1, 2, \ldots, n) \):
\[
a \in G_I(n) = G(n) \quad \text{and} \quad ((\psi_I)^{-1} \circ \psi_I)(a) = id(a) = a.
\]

Symmetry. Let \( a, b \) be elements from \( IL(\mathbb{G}) \) and \( a \equiv b \). Then there exist index tuples \( I, J \) of the same cardinality \( n \):
\[
a \in G_I(n), \quad b \in G_J(n) \quad \text{and} \quad ((\psi_I)^{-1} \circ \psi_J)(a) = b.
\]

From the definition of \( \equiv \) we obtain
\[
a = ((\psi_I)^{-1} \circ \psi_J)^{-1}(b)
\]

Then the equality
\[
((\psi_I)^{-1} \circ \psi_J)^{-1}(b) = ((\psi_J)^{-1} \circ \psi_I)(b),
\]

implies the equality
\[
((\psi_J)^{-1} \circ \psi_I)(b) = a.
\]

**Transitivity.** Let \( a, b, c \in IL(\mathbb{G}) \) be such that \( a \equiv b \) and \( b \equiv c \). From the definition of \( \equiv \) it follows that there exist index tuples \( I, J_1, J_2 \) of cardinality \( n_1 \) such that
\[
a \in G_I(n_1), b \in G_{J_1}(n_1), ((\psi_I)^{-1} \circ \psi_{J_1})(a) = b,
\]

and also exist index tuples \( J_2, K \) of cardinality \( n_2 \) such that
\[
b \in G_{J_2}(n_2), c \in G_K(n_2), ((\psi_{J_2})^{-1} \circ \psi_K)(b) = c.
\]

Denote by \( m \) the cardinality \( |J_1 \cap J_2| \).

Let
\[
A = (\max i + 1, \ldots, \max i + n_2 - m),
\]
\[
B = (\max k + 1, \ldots, \max k + n_1 - m)
\]
and define the following index tuples:
\[
\mathcal{T} = I \cup A,
\]
\[
\mathcal{T}_1 = J_1 \cup (J_2 \setminus J_1),
\]
\[
\mathcal{K} = K \cup B,
\]
\[
\mathcal{T}_2 = J_2 \cup (J_1 \setminus J_2).
\]

Denote by \( N \) the sum \( n_1 + n_2 - m \). Then \( |\mathcal{T}| = |\mathcal{T}_1| = |\mathcal{K}| = |\mathcal{T}_2| = N \).

Denote by \( g_1 \) the element \( (\psi_I)^{-1}(a) \). Then \( g_1 \in G(n_1) \). Equality (2) implies that \( g_1 = (\psi_{J_1})^{-1}(b) \).

Since \( \mathcal{T} = I \cup A \) the inclusion \( SoG_{\mathcal{T}} \supset SoG_I(G) \) holds. It implies that \( G_{\mathcal{T}}(N) > G_I(n_1) \). Hence, we obtain
\[
(\psi_{\mathcal{T}})^{-1}(a) = (\psi_I)^{-1}(a) = g_1.
\]

Similarly, from the equality \( \mathcal{T}_1 = J_1 \cup B \) we obtain:
\[
(\psi_{\mathcal{T}_1})^{-1}(b) = (\psi_{J_1})^{-1}(b) = g_1.
\]

Denote by \( g_2 \) the element \( (\psi_K)^{-1}(c) \). Equation (3) implies that \( g_2 \in G(n_2) \). Similar to the previous case one can show that
1. \( (\psi_{\mathcal{T}_2})^{-1}(b) = (\psi_{J_2})^{-1}(b) = g_2 \).
2. \( (\psi_{\mathcal{K}})^{-1}(c) = (\psi_K)^{-1}(c) = g_2 \).

Since index tuples \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) contains the same numbers, we have the equality \( G_{\mathcal{T}_1}(N) = G_{\mathcal{T}_2}(N) \). Then the mapping
\[
(\psi_{\mathcal{T}})^{-1} \circ \psi_{\mathcal{T}} : G_{\mathcal{T}}(N) \rightarrow G_{\mathcal{T}}(N)
\]
maps \( a \) to \( b \) and the mapping
\[
(\psi_{\mathcal{T}_1})^{-1} \circ \psi_{\mathcal{T}_1} : G_{\mathcal{T}_1}(N) \rightarrow G_{\mathcal{T}_1}(N)
\]
maps \( b \) to \( c \).

It follows that the composition
\[
(\psi_{\mathcal{T}})^{-1} \circ \psi_{\mathcal{T}} \circ (\psi_{\mathcal{T}_1})^{-1} \circ \psi_{\mathcal{T}_1} : G_{\mathcal{T}}(N) \rightarrow G_{\mathcal{T}}(N)
\]
maps \( a \) to \( c \).

We are left to show that the composition (6) can be re-combined so that it is a product of two isomorphisms, according to the definition of \( \cong \). It is enough to show that there exists an index tuple \( I \) of cardinality \( N \) such that:

\[
(\psi_I)^{-1} = (\psi_J)^{-1} \circ \psi_{\bar{J}} \circ (\psi_{\bar{J}})^{-1}.
\]

Assume that \( T = (i_1, \ldots, i_N) \). Note that \( T_1 = T_2 \). Hence, there exists a permutation \( \pi : 1, N \to 1, N \) such that:

1. \( T_1 = (j_{\pi(1)}, \ldots, j_{\pi(N)}) \);
2. \( T_2 = (j_1, \ldots, j_N) \).

Let \( s \) be arbitrary generator from \( S_{\Omega(G)}(N) \). Then its index is \( t = (k - 1) \cdot C + r \) for some \( k \in 1, N \), \( r \in 1, C \). Then from (1) we obtain:

\[
(h_{T_2}(t) = (j_k - 1) \cdot C + r = (j_{\pi(t)} - 1) \cdot C + r,
\]

\[
(h_{T_2})^{-1}((j_k - 1) \cdot C + r) = (t - 1) \cdot C + r,
\]

\[
(h_{T_2})^{-1}(1 - 1) \cdot C + r = (i_k - 1) \cdot C + r,
\]

where \( t \) is the position of \( j_k \) in \( T_2 \) which is mapped to \( j_k \) by \( \pi \). Define the index tuple \( \bar{I} := (i_k|\pi(t) = k, k \geq 1, \) Then

\[\psi_I(s) = ((\psi_{\bar{J}})^{-1} \circ (\psi_{\bar{J}})^{-1} \circ \psi_{T_2})(s).\]

From the definition of \( \cong \) we obtain \( (\psi_{\bar{T}})^{-1} \circ (\psi_{\bar{T}})^{-1}(a) = c \).

The proof is complete.

**Definition 10.** Elements \( a, b \in IL(G) \) are called homogeneously equivalent if \( a \cong b \).

**Definition 11.** The homogeneous class of an element \( a \in IL(G) \) is the subset of all elements from \( IL(G) \), which are homogeneously equivalent to \( a \):

\[HC(a) = \{ b \in IL(G) | a \cong b \} \]}

**Properties of a homogeneous class.**

**Lemma 15.** The set \( \{ e \} \) is the (trivial) homogeneous class of \( e \).

**Proof.** Let \( a \in G(n) \) for some natural \( n \). Suppose that \( a \in HC(e), a \neq e \). From the definition of the homogeneous equivalence we obtain \( e \cong a \). This means that there exist index tuples \( I, J \) of cardinality \( n \) such that:

\[(\psi_I)^{-1} \circ \psi_J(e) = a.\]

Note that \( \psi_I, \psi_J \) are group isomorphisms. Hence, \( (\psi_I)^{-1} \circ \psi_J \) is a group isomorphism as well. It means that:

\[(\psi_I)^{-1} \circ \psi_J(e) = e,\]

which leads to a contradiction with inequality \( a \neq e \).

The proof is complete.

**Lemma 16.** Let \( a \in G_I(n), b \in G_J(n) \) and \( a \cong b \) for some index tuples \( I, J \) of cardinality \( n \). Then

\[|a|_{S_{\Omega(G_I)}(n)} = |b|_{S_{\Omega(G_J)}(n)}.\]

**Proof.** Denote by \( l \) the length \( l \). Suppose that

\[|a|_{S_{\Omega(G_I)}(n)} > l.\]

Then there exist generators \( s_{j_1}, s_{j_2}, \ldots, s_{j_k} \in S_{\Omega(G_J)}(n) \) such that \( b = \prod_{k=1}^{l} s_{j_k} \).

Since the groups-generators series \( G \) is homogeneous the decomposition

\[\prod_{k=1}^{l} (\psi_I)^{-1} \circ \psi_J(s_{j_k}) = \prod_{k=1}^{l} (\psi_I)^{-1} \circ \psi_J(s_{j_k}) = a\]

is a decomposition of the element \( a \) over \( S_{\Omega(G)}(n) \). Hence, \( |a|_{S_{\Omega(G_I)}(n)} \leq l \). A contradiction.

Similarly the assumption \( |a|_{S_{\Omega(G_I)}(n)} < l \) leads to a contradiction.

The proof is complete.

Lemma 16 gives rise to the following definition.

Let \( HC \) be a homogeneous class such that its intersection with \( G \) is non-trivial.

**Definition 12.** A length of the homogeneous class \( HC \) over \( S \) is defined as:

\[|HC|_S = |a|_S,\]

where \( a \) is an element from \( HC \cap G \).

**Lemma 17.** Let \( a, b \in G(n), a \cong b \) for some natural \( n \). Then there exists an automorphism \( \psi \) of \( G(n) \) such that:

\[\psi(a) = b,\]

whose restriction on \( S_{\Omega(G)}(n) \) is a permutation.

**Proof.** From \( a \cong b \) it follows that for some index tuples \( I, J \) the mapping \( \psi := (\psi_I)^{-1} \circ \psi_J \) is an automorphism of \( G(n) \) such that:

\[\psi(a) = b.\]

From the definition of mappings \( \psi_I, \psi_J \) it follows that the composition \( \psi \) is a permutation on \( S_{\Omega(G)}(n) \).

**Lemma 18.** Let \( HC \) be a homogeneous class and \( a \) be a properly generated element from \( HC \cap G(n) \) over \( S_{\Omega(G)}(n) \) for some natural \( n \). Then every element of \( HC \cap G(n) \) is properly generated over \( S_{\Omega(G)}(n) \).
Therefore, there exists a permutation \( \pi \) of \( H \cap G(n) \). Suppose that \( b \) is not properly generated over \( S \cup G(n) \). Then there exists a decomposition \( D = (s_1, \ldots, s_n) \) of \( b \) over \( S \cup G(n) \) such that \( S \cup G(n) \setminus D \neq \emptyset \) as sets. Note that \( a, b \) belong to the same group \( G(n) \). Since elements from the same group both index \( H \cup G \) and \( n \) elements, \( \psi \exists \) an automorphism \( \psi \) of \( G(n) \) such that
\[
\psi(b) = a.
\]
The homogeneous property of groups-generators series \( G \) implies that \( \psi(D) = (\psi(s_1), \ldots, \psi(s_n)) \) is a decomposition of the element \( a \) over \( S \cup G(n) \). Moreover, the restriction of \( \psi \) on \( S \cup G(n) \) is a permutation. Hence, \( S \cup G(n) \setminus \psi(D) \neq \emptyset \) as sets. It means, that the element \( a \) is not properly generated over \( S \cup G(n) \). A contradiction.

The proof is complete.

**Proposition 19.** Let \( HC \) be a homogeneous class and \( a \in HC \cap G(n) \) for some natural \( n \). If \( a \) is a diameter element of \( G(n) \), then every element of \( HC \cap G(n) \) is a diameter element of \( G(n) \).

**Proof.** Directly implies from the previous lemma.

**Homogeneous down search algorithm**

Let \( G \) be a homogeneous groups-generators series, \( n \) be a natural number. Assume that \( G = G(n), S = SoG(n) \).

Let \( HC \) be a homogeneous class such that \( HC \cap G \neq \emptyset \). Fix an element \( hc \in HC \cap G \). We define the product
\[
HC \cdot S = \{HC(hc \cdot s)|s \in S\}, \tag{7}
\]

**Lemma 20.** The product (7) of the homogeneous class \( HC \) and the system of generators \( S \) is well defined.

**Proof.** Let \( h_{c_1}, h_{c_2} \) be different elements from \( HC \cap G \). Lemma 17 for \( h_{c_1}, h_{c_2} \) states that there exists an automorphism \( \psi \) of \( G \) such that:
\[
\psi(h_{c_1}) = h_{c_2}.
\]

Since elements from the same group both index tuples \( I \) and \( J \) consist of numbers \( \{1, 2, \ldots, n\} \). Therefore, there exists a permutation \( \pi : I \cdot C \rightarrow J \cdot C \) such that:
\[
\psi(s_i) = s_{\pi(i)}
\]
for every \( i \in \Gamma \cdot n \cdot C \). Note that \( n \cdot C = |S| \).

Let \( i, j \in \Gamma \cdot |S| \) be indices of generators in \( S \)
\[
\pi(i) = j.
\]
Then
\[
h_{c_2} \cdot s_j = h_{c_2} \cdot s_{\pi(i)} = \psi(h_{c_1}) \cdot s_i = \psi(h_{c_1} \cdot s_i).
\]

The definition of homogeneous equivalence implies that \( h_{c_2} \cdot s_j \in HC(h_{c_1} \cdot s_i) \). Hence, for every generator \( s_j \in S \) there exists unique \( s_i \in S \) such that
\[
h_{c_2} \cdot s_j \in HC(h_{c_1} \cdot s_i).
\]

The definition of homogeneous equivalent now implies the equality
\[
HC(h_{c_2} \cdot s_j) = HC(h_{c_1} \cdot s_i).
\]

Then moving through all generators of \( S \) we have:
\[
\{HC(h_{c_2} \cdot s)\mid s \in S\} = \{HC(h_{c_1} \cdot s)\mid s \in S\}.
\]

The proof is complete.

**Algorithm 4: Homogeneous down search algorithm**

**Input:** \( G \) — a group, \( S \) — its system of generators

**Result:** Diameter \( D_\Sigma(G) \)

**Initialization:** \( \) found = \( \{HC(e)\} \),
\( all = \{HC(g)|g \in G\} \),
\( current\_level = \{HC(e)\}, level = 0 \).

**while** \( \) found \( \neq \) all **do**
\( previous\_level, current\_level = current\_level, \{\} \);
\( for \) \( HC \in previous\_level \) **do**
\( current\_level = current\_level \cup HC \cdot S; \)
\( end \)
\( current\_level = current\_level \cup found; \)
\( found = found \cup current\_level; \)
\( level = level + 1; \)
\( end \)

**Output:** level

**Lemma 21.** Let \( a \in G, |a| = l \). Then \( l \) is the number of iterations of the main loop of homogeneous down search algorithm required to obtain the homogeneous class \( HC(a) \).

**Proof.** Induction on \( l \).

The basis: case \( l = 1 \). Note that on the initialization phase of the algorithm we have equalities
\[
current\_level = found = \{HC(e)\}.
\]

Hence, when \( level = 1 \) and \( CL = current\_level \) the following set of homogeneous classes will appear:
\[
(\bigcup_{HC \in CL} HC \cdot S) \setminus found = (HC(e) \cdot S) \setminus \{HC(e)\} = \{HC(e)\cdot s \mid s \in S\} \setminus \{HC(e)\} = \{HC(s) \mid s \in S\}.
\]

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Let \(|a|_S = 1\). Then there exists \(i \in \Gamma,|S|\) such that \(a = s_i\). From previous equalities for \(current\_level\) it follows that the class \(HC(s_i)\) will appear on the first iteration of the main loop.

From the other hand, let \(HC(a)\) appears on the first iteration of the main loop. Then, from previous equalities for \(current\_level\) it follows that there exists \(s \in S\) such that \(HC(s)\) appears as a product \(e \cdot s\) on the first step of the main loop and equality \(HC(s) = HC(a)\) holds.

Lemma 17 implies that there exists an automorphism \(\psi\) of \(G\) such that:

\[
\psi(s) = a.
\]

Note that by the same Lemma, \(\psi\) is a permutation on \(S\). Then there exists \(j \in \Gamma,|S|\) such that \(\psi(s) = s_j = a\). Hence, \(|a|_S = |s_j|_S = 1\).

**Inductive step:** case \(l+1\) under assumption that for \(l\) the statement holds.

Let \(|a|_S = l + 1\). Then there exist \(i_1, i_2, \ldots, i_{l+1} \in \Gamma,|S|\) such that:

\[
a = \prod_{k=1}^{l+1} s_{i_k}.
\]

Then the element \(b = \prod_{k=1}^{l} s_{i_k}\) has length \(l\). Otherwise, the length of \(a\) over \(S\) is less than \(l+1\).

Then, by inductive assumption, \(HC(b)\) appears on the \(l\)th step of the algorithm. Lemma 21 implies that \(HC(a) = HC(b \cdot s_{i_{l+1}})\). Then \(HC(a)\) appears on the \((l+1)\)th iteration of the algorithm. Otherwise, the element \(a\) appears on the same previous level. It leads to a contradiction with inductive assumption.

Let \(HC(a)\) appears on the \((l+1)\)th step of the algorithm. Then for some \(b \in G\) and \(s \in S\) we have the equality

\[
HC(a) = HC(b \cdot s).
\]

The inductive assumption implies \(|b|_S = l\). The last equality leads to equality \(b \cdot s = (\prod_{k=1}^{l} s_{i_k}) \cdot s\) for some \(s_{i_k} \in S\). This decomposition is minimal for \(b \cdot s\). Otherwise, \(HC(b \cdot s) = HC(a)\) appears earlier than on \((l+1)\)th level. Therefore \(|a|_S = |HC|_S = l + 1\).

The proof is complete.

**Corollary 22.** Let \(HC\) be a homogeneous class and \(HC\) appears on the \(l\)th step of homogeneous down search algorithm for \(G\) and \(S\). Then \(|el|_S = l\) for every \(el \in HC \cap G\).

**Proof.** Let \(a \in HC \cap G\). Lemma 21 implies that if \(HC\) appears on the \(l\)th step of the algorithm then \(|a|_S = l\).

The proof is complete.

**Theorem 23.** Homogeneous down search algorithm is correct.

**Proof.** The algorithm terminates if and only if \(found = all\). This equality holds if and only if every homogeneous class with non-trivial intersection with \(G\) appears at least once. This statement follows from Lemma 21 and existence of the minimum decomposition for every element.

Moreover, the last level of the algorithm contains homogeneous classes of elements of \(G\), which have the maximum length over \(S\). It means that if algorithm stops on step \(l\), then from Corollary 22 it follows

\[
|el|_S = l = D_S(G)
\]

for every \(HC \in last\_level\) and every \(el \in HC \cap G\).

The proof is complete.

**Homogeneous middle down search algorithm**

Let \(G\) be a homogeneous groups-generators series, \(n\) be a natural number. Assume that \(G = G(n), S = S \cap G(n)\).

Let \(HC\) be a homogeneous class with non-trivial intersection with \(G\). Recall that

\[
HC \cdot S = \{HC(hc \cdot s)|s \in S\},
\]

where \(hc\) is a fixed element from the intersection \(HC \cap G\).

We will use the following notations:

1. \(G_{fh}\) is the set of all properly generated homogeneous classes of the group \(G\) over \(S\).
2. \(D_{fh}(S, m)\) is the set of all decompositions over \(S\) of length \(m\) such that the following property holds:

if some element of a homogeneous class has a decomposition of length \(m\) then \(D_{fh}(S, m)\) contains at least one decomposition of length \(m\) of some element of this homogeneous class, i.e

if \(D \in D_{f}(S, m)\), \(P(D) \in HC\) then

there exists \(DH \in D_{fh}(S, m)\)

such that \(P(DH) \in HC\).
Algorithm 5: Homogeneous middle down search algorithm

Input: $G$ — a group, $S$ — its strictly growing system of generators
Result: Diameter $D_s(G)$

Initialization: $\text{found} = \emptyset$, all $= G_{fh}$, $\text{level} = |S| - 1$

while $\text{found} \neq \text{all}$ do
  $\text{level} = \text{level} + 1$
  for $\text{decomp} \in D_{fh}(S, \text{level})$ do
    $\text{product} = HC(P(\text{decomp}))$
    if $\text{product} \in G_{fh}$ then
      $\text{found} = \text{found} \cup \text{product}$
    end
  end
end

Output: $\text{level}$

Theorem 24. The homogeneous middle down search algorithm is correct.

Proof. Let $m$ be the iteration of homogeneous middle down search algorithm when $\text{found} = \text{all}$. Let $D_S(G) = l$ for some natural $l$.

Suppose that $a \in G$ is a diameter element of $G$. Then strictly growing property implies that the element $a$ is properly generated. Lemma 10 implies that the length of $a$ over $S$ is not less than $|S|$. It follows that the minimum decomposition of the element $a$ belongs to $D_f(S, l)$. Homogeneous property implies that $D_{fh}(S, l)$ contains a decomposition defining a product homogeneously equivalent to $a$. And this decomposition has the same length $l$. Therefore, we have the inequality

$$m \geq l.$$ 

Let $HC$ be a homogeneous class. Suppose that $HC$ is found on step greater than $l$. It follows that there is no decomposition of $HC$ with length from $|S|$ to $l$. Lemma 16 implies that there is no decomposition of any element from $HC \cap G$ with length from $|S|$ to $l$. But the diameter element of $G$ has length $l$. It means that there exists a decomposition of an element from $HC \cap G$ of length strictly less than $|S|$. From Lemma 10 it follows that every element of $HC \cap G$ is not properly generated. Hence, $HC$ is not in $G_{fh}$. We obtain the inequality

$$m \leq l.$$ 

Therefore, we have the equality $m = l$.

The proof is complete.

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АЛГОРИТМИ ПОШУКУ ДІАМЕТРА ОРІЄНТОВANNИХ ГРАФІВ КЕЛІ

Розглянуто дієву відому задачу пошуку діаметра скінченної групи. Вона формулюється так: знайти найбільший серед діаметрів групи відстань її систем твірних. Діаметр групи є діаметром графа Келі, що будуть на основі графів та її систем твірних. У цій роботі розглянуто підзадачі задачи пошуку діаметра групи, а саме, задачу знаходження діаметра групи відносно заданої системи твірних. Показано, що ця задача поліноміально зводиться до задачі пошуку мінімальних розкладів елементів.

Для розв’язання задачі знаходження діаметра групи відносно заданої системи твірних запропоновано п’ять алгоритмів: простий алгоритм пошуку відстаней, швидкий алгоритм пошуку відстаней, середній алгоритм пошуку відстаней, однорідний алгоритм пошуку відстаней та однорідний середній алгоритм пошуку відстаней.

Перші два алгоритми є універсальними, а інші вимагають виконання додаткових умов на
Для алгоритму середнього спуску введено поняття строго зростаючої системи твірних. За виконання цієї умови, пошук мінімальних розкладів потенційних найдовших розкладів можна почати одразу ж із певної множини.

Введено відмінну теорему однорідності. В ній розглядається ряд груп та їх систем твірних, що задовольняють певним додатковим умовам. Введено властивість однорідності таких рядів та відношення еквівалентності їх елементів. Основною метою введення такого відношення є збереження розкладів її елементів в одному класі. Ця властивість дає можливість структуризувати мінімальний розклад лише для представників класу еквівалентності.

Для алгоритмів однорідного пошуку вказати однорідного середнього пошуку вказати додатковим умовам застосування еквівалентності групи до однорідного генеративного ряду груп. Тоді задача знаходження мінімальних розкладів елементів зводиться до знаходження мінімальних розкладів представників класів еквівалентності.

Показано, що всі описані алгоритми є коректними. Зроблено оцінки складності їх роботи.

Ключові слова: граф Келі, діаметр групи, система твірних.