УДК 510.22

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DOI: 10.18523/2617-7080520223-5

REMARKS ON MY ALGEBRAIC PROBLEM OF DETERMINING SIMILARITIES BETWEEN CERTAIN QUOTIENT BOOLEAN ALGEBRAS

Remarks on my algebraic problem of determining similarities between certain quotient boolean algebras.

In this paper we survey results about quotient boolean algebras of type $\mathcal{P}(\kappa)/\text{fin}(\kappa)$ and condition for them to be or not to be isomorphic for different cardinals κ. Our consideration have their root in the classical result of Parovicenko and a less classical, nevertheless really considerable result about non-existence of P-points by S Shellah. Our main point of interest are the algebras $\mathcal{P}(\omega)/\text{fin}(\omega)$ and $\mathcal{P}(\aleph_1)/\text{fin}(\aleph_1)$.

Keywords: logic, boolean algebras, forcing.

By $\omega = \aleph_0$ we will denote the set of natural numbers. For any set X by fin(X) we will denote the family of all finite subsets of X.

Definition 1. *By a boolean algebra we will mean a set A with at least two distinct elements 0 and 1, endowed with binary operations $+$ and $\cdot$ and a unary operation $-$ satisfying the following properties:*

- *both $(B, +, 0)$ and $(B, \cdot, 1)$ are commutative monoids,*
- *$+$ is distributive with respect to $\cdot$,*
- *$\cdot$ is distributive with respect to $+$,*
- $\forall_{a,b \in A} a + (a \cdot b) = a$,
- $\forall_{a,b \in A} a \cdot (a + b) = a$,
- $\forall_{a \in A} a + (-a) = 1$,
- $\forall_{a \in A} a \cdot (-a) = 0$.

In any boolean algebra A one can introduce partial ordering by putting $a \leq b \Leftrightarrow a + b = b$. One of the most popular examples of boolean algebras are $\mathcal{P}(X)$ with $\emptyset, X, \cup, \cap$ for any non-empty set X.

Definition 2. *Let A be a boolean algebra. We will say that $I \subset A$ is an ideal in A if $0 \in I$, $1 \notin I$, it is closed under $+$ and for any $a \in I$ and $b \leq a$ we have $b \in I$. We can define am equivalence relation on A by*

$$a\ b \Leftrightarrow a \triangle b \in I$$

where $a \triangle b = (a \cdot (-b)) + (b \cdot (-a))$ and consequently we can define a quotient algebra A/I as the family of equivalence classes of with operations extending in a clear way.

Observe that fin(X) is an ideal in $\mathcal{P}(X)$.

There is a very well know theorem by Parovicenko concerning universal algebra, model theory and topology.

Definition 3. *Let A be a boolean algebra. A gap in A of type (κ, λ) will be a pair (L, R) of sequences in A such that*

- *$|L| = \kappa$ and L is increasing,*
- *$|R| = \lambda$ and R is decreasing,*
- *$l \leq r$ for any $l \in L$ and $r \in R$.*

A gap is said to be filled if there exists $c \in A$ satisfying $l \leq c \leq r$ for any $l \in L$ and $r \in R$. Otherwise a gap is said to be unfilled.

Definition 4. *Let A be a boolean algebra. A limit in A of length λ will be a sequence $s \colon \lambda \to A$ such that*

- *s is increasing,*
- *s is unbounded.*

Theorem 1. *Under assumption of CH (the Continuum Hypothesis) any topological space X such that:*

- *X is compact Hausdorff*
- *X is dense in itself*
- *the weight of X - ie the minimal cardinality of a base for its topology - is exactly $\mathfrak{c}$*
- *disjoint open F_σ sets in X have disjoint closures*
- *non-empty G_δ sets have non-empty interior*

is homeomorphic to the space $\omega^* = \beta\omega \setminus \omega$, ie to the remainder of the Cech-Stone compactification of natural numbers. [8][1]

The theorem above can be rephrased in terms of boolean algebras in a following way. Both ways of phrasing the theorem are in direct correspondence by taking the stone space of a boolean algebra as a topological space and by taking the algebra of all clopen subsets of a topological space as a boolean algebra.

Theorem 2. *Under assumption of CH any boolean algebra A such that:*

- $|A| = \mathfrak{c}$,

- *A is atomless,*
- *A has no limits of length* ω*,*
- *A has no gaps of type* (ω, ω)

is isomorphic to the quotient algebra $\mathcal{P}(\omega)/\mathrm{fin}(\omega)$*.*

It has been proved in 1980s independently by me [4] as well as Van Mill and Van Douven [5] that this result is not only a consequence of CH but is in fact equivalent to it. During a proof of such an equivalence a problem of determining similarities between the boolean algebras $\mathcal{P}(\kappa)/\mathrm{fin}(\kappa)$ for different cardinals κ naturally occurs. In [6] together with Balcar we have shown that for $\omega \leq \lambda < \kappa$ and $\kappa \geq \aleph_2$ the algebras $\mathcal{P}(\kappa)/\mathrm{fin}(\kappa)$ and $\mathcal{P}(\lambda)/\mathrm{fin}(\lambda)$ are not isomorphic. The proof for that is based on the following theorem.

Theorem 3. *If* $\mathcal{P}(\omega)/\mathrm{fin}(\omega)$ *and* $\mathcal{P}(\aleph_1)/\mathrm{fin}(\aleph_1)$ *then there exists a scale of length* $\aleph_1$ *in* ω^ω*, ie there exist* $S \subseteq \omega^\omega$*, such that* $|S| = \aleph_1$ *and for any* $f : \omega \to \omega$ *there exist* $g \in S$ *such that* $g(n) > > f(n)$ *for all but finitely many* $n \in \omega$*.*

The notion of scale has been introduced by F Hausdorff in [9]. As of now the problem in all its generality whether it is equiconsistent with ZFC that the algebras $\mathcal{P}(\omega)/\mathrm{fin}(\omega)$ and $\mathcal{P}(\aleph_1)/\mathrm{fin}(\aleph_1)$ are isomorphic (ie under assumption of existence of a model for ZFC can they be isomorphic in some model) remains open.

The next breakthrough came in [7] when together with P Zbierski and M Grzech we showed that it is equiconsistent with ZFC that $\mathfrak{c} = \aleph_2$ and the completions of the algebras $\mathcal{P}(\omega)/\mathrm{fin}(\omega)$ and $\mathcal{P}(\aleph_1)/\mathrm{fin}(\aleph_1)$ are isomorphic. More precisely the following holds.

Definition 5. *Let* X *be a topological space and* $x \in X$*. We will say* x *is a P-point if for any open sets* $U_i \subseteq$ *for* $i \in \omega$ *such that* $x \in U_i$ *there exists an open set* $U \subseteq X$ *such that*

$$x \in U \subseteq \bigcap_{i \in \omega} U_i.$$

Similarly a set $A \subseteq X$ *will be called a P-set if for any open sets* $U_i \subseteq$ *for* $i \in \omega$ *such that* $A \subseteq U_i$ *there exists an open set* $U \subseteq X$ *such that*

$$A \subseteq U \subseteq \bigcap_{i \in \omega} U_i.$$

Definition 6. *Let* X *be a topological space,* κ *be an uncountable cardinal and* $U \subseteq X$*. We will say that* U *has the* κ*-cc (antichain condition) if any family of pairwise disjoint, non-empty subsets of* U *has the cardinality strictly less than* κ*.*

If U *has* $\aleph_1$*-cc then we will say that it has ccc (countable antichain condition).*

The corresponding definition can be made for antichains in boolean algebras.

Theorem 4. *If* G *is a generic ultrafilter of Grigorieff forcing then in the model* $V[G]$ *there are no P-sets that satisfy* $\mathfrak{c}$*-cc.*

Theorem 5. *If* G *is a generic ultrafilter of Grigorieff forcing then in the model* $V^{\mathbb{P}_{\omega_2}}[G]$ *every fat P-set* F *has a* π*-base tree of height* ω*, each vertex of which splits into* $\mathfrak{c}$ *elements.*

In an upcoming work by replacing the Grigorieff forcing by a more refined forcing notion we will be able to show that the problem whether $\mathcal{P}(\omega)/\mathrm{fin}(\omega)$ and $\mathcal{P}(\aleph_1)/\mathrm{fin}(\aleph_1)$ are isomorphic is in fact equivalent to the existence of a special type of partitioners in the algebra $\mathcal{P}(\omega)/\mathrm{fin}(\omega)$.

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ЗАУВАЖЕННЯ ЩОДО МОЄЇ АЛГЕБРАЇЧНОЇ ПРОБЛЕМИ ВИЗНАЧЕННЯ ПОДІБНОСТІ МІЖ ДЕЯКИМИ ФАКТОРНИМИ БУЛЕВИМИ АЛГЕБРАМИ

У цій статті ми розглядаємо результати щодо факторних булевих алгебр типу $\mathcal{P}(\kappa)/\mathrm{fin}(\kappa)$ *та відповідаємо на запитання, чи є булеві алгебри ізоморфними для різних кардиналів* κ. *Наші міркування беруть своє коріння з класичного результату Паровіченка і менш класичного, проте дійсно вагомого результату про відсутність* P*-точок за С.Шелах. Головна мета нашої статті — це розгляд алгебр* $\mathcal{P}(\omega)/\mathrm{fin}(\omega)$ *і* $\mathcal{P}(\aleph_1)/\mathrm{fin}(\aleph_1)$.

Ключові слова: логіка, булеві алгебри, форсування.

Матеріал надійшов 25.10.2022