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A SOLUTION OF A FINITELY DIMENSIONAL HARRINGTON PROBLEM FOR CANTOR SET

In this paper we are exploring application of fusion lemma - a result about perfect trees, having its origin in forcing theory - to some special cases of a problem proposed by Leo Harrington in a book Analytic Sets. In all generality the problem ask whether given a sequence of functions from $\mathbb{R}^\omega$ to $[0;1]$ one can find a subsequence of it that is pointwise convergent on a product of perfect subsets of $\mathbb{R}$. We restrict our attention mainly to binary functions on the Cantor set as well as outline the possible direction of generalization of result to other topological spaces and different notions of measurablity.

Keywords: perfect trees, fusion sequences .

Preliminary notions

By 2 we mean a set $\{0, 1\}$ with discrete topology. By $C \subseteq \mathbb{R}$ we mean a standard Cantor set. It is well known that C is homeomorphic to the space 2^ω with product topology where ω denotes the set of all natural numbers.

In [1] in the problems section L Harrington published a following problem (as a possible weakening of a problem of Halpern).

Problem 1. *Given continuous functions $f_n \colon \mathbb{R}^\omega \to [0;1]$, do there exist a set $N = \{n_i \colon i \in \omega\} \in [\omega]^\omega$ and nonempty perfect sets $P_j \subseteq C$ for $j \in \omega$ such that the subsequence $(f_{n_i})_{i\in\omega}$ is pointwise convergent on the product $\prod_{j\in\omega} P_j$?*

We have acquired information that the one dimensional version of such problem has been solved in 1920s by S Mazurkiewicz, but unfortunately we were not able to trace it back to the original paper. In [2] Laver showed amongst others that we get an equivalent problem if we substitute continuous for measurable functions or functions with the Baire property. This prompted to consider variants of the problem with different notions of measurability and different topologies (most notably with Ellentuck topology [3]).

We will focus on the finitely dimensional version of the problem for binary functions on C and explore applying to it the fusion lemma - a result about perfect trees, that originated in forcing theory.

Definition 1. *Let $T \subseteq 2^{<\omega} = \bigcup_{n\in\omega} 2^n$.*

We will say that T is a tree if for any $s \in T$ and $t \in 2^{<\omega}$ if $t \subseteq s$ then $t \in T$, ie it is closed under taking the initial segment.

We will further say that T is a perfect tree if for any $s \in T$ there exist $t_1, t_2 \in T$ with $t_1 \neq t_2$ such that $s \subset t_1$ and $s \subset t_2$.

Definition 2. *Let T be a perfect tree and for each $s \in T$ let there be a perfect set $U_s \subseteq \mathbb{R}$. Then $(U_s)_{s\in T}$ is called a fusion sequence if*

- *$U_{s_1} \supseteq U_{s_2}$ for $s_1, s_2 \in T$ and $s_1 \subseteq s_2$,*
- *$U_{s\_0} \cap U_{s\_1} = \emptyset$ for $s \in T$.*

We have a following fundamental property of fusion sequences.

Theorem 1. *(Fusion Lemma) Let T be a perfect tree and let $\tilde{T} = \{f \in 2^\omega \colon \forall_{n\in\omega} f|_n \in T\}$. Let $(U_s)_{s\in T}$ be a fusion sequence. If the diameter of U_s tends to 0 with increasing length of s then the set*

$$P = \bigcap_{n\in\omega} \bigcup_{s\in T} U_s = \bigcup_{f\in\tilde{T}} \bigcap_{n\in\omega} U_{f|n}$$

is a perfect set. [6][7]

Theorem 2. *Let $P \subseteq C$ be perfect and non-empty. Then P is homeomorphic to C.*

Main result

We will apply fusion lemma to our problem.

Theorem 3. *Let $f_n \colon C \to 2$ be continuous functions. Then there exists $N = \{n_i \colon i \in \omega\} \in [\omega]^\omega$ and a non-empty perfect set $P \subseteq C$ such that the subsequence $(f_{n_i})_{i\in\omega}$ is pointwise convergent on P. Moreover the set P can chosen in such a way that it is homeomorphic to C.*

Outline of proof: Let $U_i^n = f_n^{-1}(i)$. Observe that those are perfect sets. If there exists a perfect set P such that $P \cap U_0^n = \emptyset$ or $P \cap U_1^n = \emptyset$ for infinitely many n then it easy to find a subsequence with required result. Thus we can assume - taking a subsequence if needed - that all U_i^n are non-empty and for any non-empty perfect set P we have $P \cap U_0^n \neq \emptyset$ and $P \cap U_1^n \neq \emptyset$. Now for any finite sequence $(i_0, \ldots, i_{n-1}) \in {}^n2$ let

$$U_{(i_0,\ldots,i_n)} = U_{i_0}^0 \cap \ldots \cap U_{i_n}^n.$$

Observe that for finite sequences s_1, s_2 we have:

- if $s_1 \subseteq s_2$ then $U_{s_1} \supseteq U_{s_2}$,
- $U_{s_1\_0} \cap U_{s_1\_1} = \emptyset$.

It is thus a fusion sequence and the diameter of U_s tends to 0 with increasing length of s. We will define two perfect subtrees of ${}^{<\omega}2$ in a following way.

Let

$$T_0^0 = T_0^1 = \{\emptyset\}$$

and

$$T_{n+1}^i = \{s\_(j,i) \colon s \in T_n^i, j \in 2\}.$$

Clearly the trees $T^i = \bigcup_{n\in\omega} T_n^i$ are isomorphic to ${}^{<\omega}2$ and in consequence perfect. From fusion lemma we obtain that the sets

$$Q^i = \bigcap_{n\in\omega} \bigcup_{s\in T_n^i} U_s$$

are non-empty and perfect. What is more the subsequence $(f_{2n})_{n\in\omega}$ is pointwise convergent on $P = Q^0 \cup Q^1$. Indeed it is convergent to 0 on Q^0 and to 1 on Q^1.

QED

We can generalize the above result in a following way.

Theorem 4. *Let $f_n \colon C^2 \to 2$ be continuous functions. Then there exists $N = \{n_i \colon i \in \omega\} \in [\omega]^\omega$ and non-empty perfect sets $P_1, P_2 \subseteq C$ such that the subsequence $(f_{n_i})_{i\in\omega}$ is pointwise convergent on $P_1 \times P_2$.*

Outline of proof: Let $D = \{c_n \colon n \in \omega\}$ be a dense subset of C. Let $g_{m,n}(x) = f_n(x, c_m)$. By the previous theorem there exists N_0 and a perfect set $P_0 = Q_0^0 \cup Q_0^1$ such that $(g_{0,n})_{n\in N_0}$ is pointwise convergent on P_0. With N_m and P_m defined there exists $N_{m+1} \subseteq N_m$ and a perfect set $P_{m+1} = Q_{m+1}^0 \cup Q_{m+1}^1 \subseteq P_m$ such that $(g_{m+1,n})_{n\in N_{m+1}}$ is pointwise convergent on P_{m+1}. We will define the set $N = \{n_m \colon m \in \omega\}$ in a following way.

Let

$$n_0 = \min(N_0)$$

and

$$n_{m+1} = \min(N_{m+1} \setminus \{n_0, \ldots, n_m\}).$$

Observe that $(g_{m,n})_{n\in N}$ is convergent on P_k for $k \geq m$. It remains to show that $P_\infty = \bigcap_{m\in\omega} P_m$ has a non-empty perfect subset and the result will follow from the fact that D is dense and continuity of the functions f_n.

Of course P_∞ is closed and from the fact that $P_{m+1} \subseteq P_m$ and compactness of C it is also non-empty. Similarly to the proof of the previous theorem we can - taking an infinite subset of D if necessary - assume that

$$Q_{(i_0,\ldots,i_n)} = Q_{i_0}^0 \cap \ldots \cap Q_{i_n}^n \neq \emptyset.$$

Once again for finite sequences s_1, s_2 we have:

- if $s_1 \subseteq s_2$ then $Q_{s_1} \supseteq Q_{s_2}$,
- $Q_{s_1\_0} \cap Q_{s_1\_1} = \emptyset$.

and the diameter of U_s tends to 0 with increasing length of s. This time we can apply fusion lemma directly and we get that the set

$$P = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} Q_s \subseteq P_\infty$$

is perfect.

QED

Corrolary 1. *The space C^2 is homeomorphic to C, in fact even C^ω is homeomorphic to C. Using that fact we can easily extend the result above to any finite dimension, ie for functions $f_n \colon C^k \to 2$.*

Remark 1. *As C is a compact space in the above theorems we in fact obtain sequences of functions that are not only pointwise but also uniformly convergent.*

Further developments

Our next objective will be generalizing the result above to the infinitely dimensional variant. After that we will proceed with proving the analogous results for $[\omega]^{<\omega}$ with the Ellentuck topology [3] (which could be thought of as a topological representation of Mathias forcing) and completely ramsey measurable functions [4]. The analogs for silver forcing are worth exploring. [5]

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РОЗВ'ЯЗОК СКІНЧЕННОВИМІРНОЇ ЗАДАЧІ ХАРРІНГТОНА ДЛЯ МНОЖИНИ КАНТОРА

У цій статті ми досліджуємо застосування леми про злиття — результат про ідеальні дерева, що походить від теорії примусу — до деяких особливих випадків проблеми, запропонованої Лео Харінгтоном у книзі «Аналітичні множини». У загальному випадку проблема полягає в тому, чи можна знайти для послідовності функцій від $\mathbb{R}^\omega$ *до* $[0;1]$ *її підпослідовність, яка поточково збіжна до добутку ідеальних підмножин* $\mathbb{R}$*. Ми розглядатимемо головним чином бінарні функції на множині Кантора, а також окреслимо можливий напрямок узагальнення результату на інші топологічні простори та різні поняття вимірності.*

Ключові слова: ідеальні дерева, послідовності злиття.

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