УДК 519.1

Y. Utenko

https://doi.org/10.18523/2617-7080620233-5

PROPERTIES OF THE IDEAL-INTERSECTION GRAPH OF THE RING Z_N

In this paper we study properties of the ideal-intersection graph of the ring Z_n. The graph of ideal intersections is a simple graph in which the vertices are non-zero ideals of the ring, and two vertices (ideals) are adjacent if their intersection is also a non-zero ideal of the ring. These graphs can be referred to as the intersection scheme of equivalence classes (See: Laxman Saha, Mithun Basak Kalishankar Tiwary "Metric dimension of ideal-intersection graph of the ring Z_n" [1]).

In this article we prove that the triameter of graph is equal to six or less than six. We also describe maximal clique of the ideal-intersection graph of the ring Z_n. We prove that the chromatic number of this graph is equal to the sum of the number of elements in the zero equivalence class and the class with the largest number of element. In addition, we demonstrate that eccentricity is equal to 1 or it is equal to 2. And in the end we describe the central vertices in the ideal-intersection graph of the ring Z_n.

Keywords: ideal-intersection graph, triameter, clique, central vertices, eccentricity.

Preliminary notions

In this article, we consider simple graphs. Laxman Saha, Mithun Basak and Kalishankar Tiwary in [1] show that for a commutative ring R, ideal-intersection graph is a simple graph with vertices that are non-zero proper ideals of the ring R and two vertices (ideals) are adjacent if their intersection is also a non-zero (proper) ideal of R. Throughout this paper, $G(R)$ denotes ideal-intersection graph of the ring R. For a given positive integer n, define $P(n)$ be the set of all primes that are used to decompose the integer n. So that for any positive integer

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r}$$

of the ring Z_n is of the forms

$$I = < p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot \ldots \cdot p_r^{\beta_r} >,$$

where β_i is are integers with $0 \leq \beta_i \leq \alpha_i$. Also they prove that for the graph $G(Z_n)$ the following properties hold:

1) Diameter of the graph $G(Z_n) \leq 2$.

2) Every distinct pair of classes preserve the distances, i.e., for two ideals $I_1 \in [I]$ and $J_1 \in [J]$, $d(I_1, J_1) = 1$ or 2 according as $d(I, J) = 1$ and $d(I, J) = 2$.

3) Each ideal class $[I]$ forms a clique in $G(Z_n)$.

It is also noticed, that for any ideal $I = \langle p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot \ldots \cdot p_r^{\beta_r} \rangle$ in the ring $\mathbb{Z}_n$ with $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r}$, we define the set $L(I) = \{p_i : \beta_i = \alpha_i\}$, which is called the set of levels for I. We also define the set $L'(I) = \{p_i : \beta_i \neq \alpha_i\}$. Clearly, for the ideal I, $L(I) \subset P(n)$, and $L'(I)$ is the complement of $L(I)$ in $P(n)$, so $L(I) \cup L'(I) = P(n)$ and $L(I) \cap L'(I) = \emptyset$. [?]

In this article we prove that the triameter of graph is equal to 6 or less than 6. We also describe clique of this graph, namely maximal clique of the ideal-intersection graph of the ring Z_n. Also we prove that the chromatic number of this graph is equal to the sum of the number of elements in the zero equivalence class and the class with the largest number of element. In addition, we demonstrate that eccentricity is equal to one or it is equal to two. And in the end we describe the central vertices in the ideal-intersection graph of the ring Z_n.

First we recall some definitions that we need.

Definition 1. [3] The eccentricity of a vertex v in an undirected graph G is defined as the maximum distance between v and any other vertex in G. In other words, it is the greatest number of edges that one must traverse to reach any other vertex in the graph from v.

The eccentricity of vertex v can be denoted as: $ecc(v) = \max\{d(v, u) : u \in G\}$

Definition 2. [3] The center of a graph is the set of central vertices, i.e., vertices with minimal eccentricity.

Definition 3. [3] A complete subgraph of a graph with the maximum amount of vertices is called the clique of the graph.

Definition 4. [2] The triameter of a simple graph G is a parameter defined as $tr(G) = \max\{d(a, b) + d(a, c) + d(b, c) : a, b, c \in V(G)\}$, where d is the shortest path between the specified vertices.

Definition 5. [7] The chromatic number of a graph G is defined as the minimum number of colors needed to color the graph in such a way that any two adjacent vertices have different colors. It is denoted as $\chi(G)$.

Main results

The next theorem characterizes the clique of the graph.

Theorem 1. *Let L_i be the ideal class of the ring $\mathbb{Z}_n$, which defines a subgraph in the intersection graph of ideals with the maximum number of vertices. Then the maximal clique of the intersection graph of ideals $\mathbb{Z}_n$ is $L_i \cup L_0$.*

Proof. Based on the previous lemma, we can assert that the complete subgraph in the ideal-intersection graph Z_n is formed by vertices of the same class. It is important to remember that an arbitrary vertex from the zero-level class is connected to any vertex. Therefore, all vertices from $L_i \cup L_0$ will form a complete subgraph and, provided that L_i contains the maximum number of vertices, it will be a maximal clique.

Theorem 2. *If there are vertices in the ideal-intersection graph of the ring $\mathbb{Z}_n$ that are not connected to each other (i.e., they belong to different non-zero equivalence classes), then $tr(G) = 6$. In the opposite case $tr(G) < 6$.*

Proof. If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$ is factored into three or more numbers, then there will be three or more equivalence classes, which means there will exist three points that are not connected to each other. They will have the maximum possible distance of one between each other, which, relying on Lemma 2.4 from [1], is equal to 2. Therefore, the sum

$$\begin{aligned} tr(G) = \max\{d_G(a,b) + d_G(a,c) + \\ + d_G(b,c) : a,b,c \in V(G)\} = \\ = tr(G) = 2+2+2 = 6. \end{aligned}$$

According to the condition, if there are no vertices that are pairwise disconnected, then the trimeter of the graph will be three or four:

1) If we have a connected graph (i.e., the graph has only one equivalence class), then the shortest distances between these vertices will be 1. Therefore, the sum

$$\begin{aligned} tr(G) = \max\{d(a,b) + d(a,c) + \\ + d(b,c) : a,b,c \in V(G)\} = \\ = tr(G) = 1+1+1 = 3. \end{aligned}$$

This holds for the example 1 for the ring Z_{16}.

2) If we have a graph where, in addition to the zero equivalence class, there are other classes, but there are no three vertices belonging to different equivalence classes, then the distance between two disconnected vertices will be 2 (the maximum possible distance). Therefore, the sum

$$\begin{aligned} tr(G) = \max\{d(a,b) + d(a,c) + \\ + d(b,c) : a,b,c \in V(G)\} = \\ = tr(G) = 1+1+2 = 4. \end{aligned}$$

Example 1. For Z_{16}: $16 = 2^4$

$P(n) = \{2\}$

$< 2 >= I_2 \to L(I_2) = \emptyset$

$< 4 >= I_4 \to L(I_4) = \emptyset$

$< 8 >= I_8 \to L(I_8) = \emptyset$

Classes of equivalence:

Class 1: I_2, I_4, I_8.

As we have only one class of equivalence, then $tr(G) = 3$.

From this theorem directly we have.

Corollary 3. *For any positive integer*

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k},$$

with $k \geq 3$, $tr(G) = 6$. Otherwise, $tr(G) < 6$.

Theorem 4. *The chromatic number of the ideal-intersection graph of the ring $\mathbb{Z}_n$ is equal to the sum of the number of elements in the zero equivalence class and the class with the largest number of elements.*

Proof. 1) Since the vertices of the zero equivalence class are connected to all vertices, including each other, it is obvious that we should color all these vertices with different colors (as they are adjacent to each other).

2) Since the vertices of one class (with the largest number of elements) are connected to each other and to the vertices of the zero class, we need to color them with different colors so that the class with fewer elements can be colored using the colors of this class (since vertices from different classes that are not zero are not adjacent).

For the other vertices, we can use the colors that are used to color the class with the largest number of vertices, as different classes, except for the zero class, have non-adjacent vertices with vertices from other classes.

Proposition 5. *Let v be a vertex in the ideal-intersection graph G of the ring $\mathbb{Z}_n$. Then the eccentricity of vertex v in graph G is equal to one if it is a vertex of the zero level and two otherwise.*

Proof. 1. Vertex of the zero level is connected with each vertex of the graph. So, path between these vertices is equal to 1. Hence, the eccentricity is equal to 1.

2. When vertex is not belong to zero level, than it belongs to the other level, vertices of which are not connected with the other vertices from non-zero class. Thus, the eccentricity is equal to the maximum possible path, namely 2.

Theorem 6. *The central vertices in the ideal-intersection graph of the ring $\mathbb{Z}_n$ are the ideals of the zero class.*

Proof. Since the vertices of the zero level are connected to all other vertices, they are vertices with the minimum eccentricity. Therefore, the ideals of the zero class are the centers.

References

1. Laxman Saha, Mithun Basak, and Kalishankar Tiwary, "Metric dimension of ideal-intersection graph of the ring Z_n AKCE," International Journal of Graphs and Combinatorics. **18** (2), 101–105 (2021).
2. Artem Hak, Sergiy Kozerenko, and Bogdana Oliynyk, "A note on the triameter of graphs," Discrete Applied Mathematics. **309**, 278–279 (2022).
3. I. Kuzmenko, *Graph theory* (2020), p. 12.
4. Andras Sebo and Eric Tannier, "On Metric Generators of Graphs", Mathematics of Operations Research. **29** (2), 383–393 (2024).
5. Dariush Kiani and Mohsen Mollahajiaghaei, "On the unitary Cayley graphs of matrix algebras", Linear Algebra and its Applications **466**, 421–428 (2015).
6. Walter Klotz and Torsten Sander, "Some properties of unitary Cayley graphs", The electronic journal of combinatorics. **14** (1), R45-R45 (2007).
7. Y. Bondarchuk and B. Olyinyk, Basics of Discrete Mathematics, in: *Graph theory*, chapter 7, pp. 111–116 (2007).
8. K. Kuratowski, *Topology*, vol. 1 (Academic Press, 1976).
9. S. Shelah, *Proper Forcing, Lecture Notes in Mathematics 940* (Springer, Berlin, 1982).

Утенко Є. О.

ВЛАСТИВОСТІ ГРАФА ПЕРЕТИНУ ІДЕАЛІВ КІЛЬЦЯ Z_N

У цій роботі ми вивчаємо властивості графа перетину ідеалів кільця Z_n. Граф перетину ідеалів є простим графом, в якому вершинами є ненульові ідеали кільця, і дві вершини (ідеали) суміжні, якщо їхній перетин також є ненульовим ідеалом кільця. Ці графи можуть бути визначені як схема перетину класів еквівалентності (див.:[1]).

У цій статті ми доводимо, що триаметр графа дорівнює шести або менше. Ми також описуємо кліку цього графа, а саме максимальну кліку графа перетину ідеалів кільця Z_n. Також ми доводимо, що хроматичне число цього графа дорівнює сумі кількості елементів у класі нульової еквівалентності та класі з найбільшою кількістю елементів. Крім того, ми демонструємо, що ексцентриситет дорівнює 1 або дорівнює 2, і наприкінці ми описуємо центральні вершини графа перетину ідеалів кільця Z_n.

Ключові слова: граф перетину ідеалів, триаметр, кліка, центральні вершини, ексцентриситет.

Матеріал надійшов 27.12.2023