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WEAKLY NONLINEAR MODELS OF STOCHASTIC WAVE PROPAGATION IN TWO-LAYER HYDRODYNAMIC SYSTEMS

The paper discusses three-dimensional models of the propagation of stochastic internal waves in hydrodynamic systems: 'half-space - half-space', 'half-space - layer with rigid lid', and 'layer with solid bottom - layer with rigid lid'. In constructing the models, the layers are considered to be ideal fluids separated by a contact surface. The main objective of the modeling is to obtain a dynamic equation for the stochastic amplitude of surface waves. A comparative analysis of the obtained results has been conducted. In order to control the contribution of nonlinear terms, a dimensionless non-numerical parameter has been introduced. The models are distinguished by boundary conditions that determine the general form of solutions. As a result, a dynamic equation for the stochastic amplitude of internal waves has been derived. After ensemble averaging of the amplitudes, the dynamic equation is formulated in integral form using Fourier-Stieltjes integrals. The dynamic equation reveals two-wave and three-wave interactions, as well as the contribution of dispersion to wave dynamics. An investigation of the boundary case of the transition of internal waves in the 'half-space - half-space' system to surface waves in the absence of an upper liquid layer confirms the validity of the results.

Keywords: stochastic waves, internal waves, wave propagation models.

Introduction

Practical interest in studying wave motions is driven by modern needs for developing new wave suppression methods, energy generation approaches, and the design of new types of water transport. Experimental and theoretical research on wave motions is conducted in many countries worldwide, including Sweden, Norway, Australia, Denmark, the USA, Japan, the UK, and China. With the advancement of modern mathematical modelling tools, particular attention is paid to the significant interest in studying random waves when addressing various applied problems, notably in hydromechanics. The relevance of researching random wave motions is also justified by the fact that such waves provide a mathematically sound approximation of real wave processes occurring on the surface and within the water layer. Investigating random waves in a two-layered fluid offers deeper insights into the mechanisms of internal wave propagation.

A brief overview of scientific research in the field of modelling stochastic wave motions in layered hydrodynamic media is presented below.

Stochastic fluid models are employed for control and optimization purposes in communication networks, particularly in admission control scenarios. The study derives gradient estimators for performance metrics related to packet loss and workload concerning these threshold parameters (G. Christos et all, 2002) [1]. The practical utility of these stochastic fluid models-based estimators is exemplified in admission control problems, using data derived from an operational system for evaluation.

Two versatile stochastic models are presented in the article (G. Lindgren et all, 2010) [2] for simulating 2D and 3D ocean waves, offering the potential to replicate extreme and spatially varying sea states. The first model encompasses generalized Lagrange models governing the motion of individual water particles. The second one is a random field model generated through a nested stochastic partial differential equation, it can be adapted to non-uniform sea conditions and offers approximations to conventional wave spectra.

The study [C.F. Naa et all, 2011) [3] introduces a novel enhancement to the moving particle semi-implicit method. The primary aim of this enhancement is to counteract energy loss attributable to numerical dissipation inherent in the conventional moving particle semi-implicit method, which leads to the rapid decay of waves. The analysis proves efficient in determining wave parameters, and it is observed that the stochastic enhancement prolongs the persistence of waves compared to the basic method.

The article (M.G. Brown, C. Lu, 2015) [4] shows that by cross-correlating time series data of seemingly random waves recorded at two different points, it becomes possible to estimate the Green's function that characterizes the wave propagation from one location to another. This study delves into the theoretical framework of random surface gravity wave interferometry, provides practical demonstrations through numerical simulations, and explores the concept experimentally using data collected from both wavetank and ocean wave measurements.

Altomare C. at all introduce a comprehensive implementation of wave generation and active wave absorption techniques for second-order long-crested monochromatic and random waves within a WCSPH-based (Weakly Compressible Smoothed Particle Hydrodynamics) model. The open-source software DualSPHysics is employed for this purpose. The second-order wave generation system, capable of producing both monochromatic (regular) and random (irregular) waves, is integrated with passive and active wave absorption mechanisms. A damping system is defined for passive absorption to prevent wave reflection from fixed boundaries within the numerical setup (C. Altomare at all, 2017) [5].

Wang Y.G. proposes a new methodology to predict the wave height and period joint distributions by utilizing a transformed linear simulation method. The proposed transformed linear simulation method is based on a Hermite transformation model where the transformation is chosen to be a monotonic cubic polynomial, calibrated such that the first four moments of the transformed model match the moments of the true process (Y.G. Wang , 2017) [6].

Utilizing the linear random wave solutions found within water wave equations applicable to finite water depths, a theoretical statistical model for the drift induced by waves is formulated by Song J., He H. and Cao A. . A straightforward scenario of a wind-generated sea is examined, and the parameters are computed for common wind velocities and water depths using the Phillips spectrum. The study explores the characteristics of the distribution and scrutinizes how variations in wind speed and water depth impact this distribution (J. Song et all, 2018) [7].

Multiscale, multi-physics uncertainty in wave-current interaction is the focus of the article of Darryl D. Holm . To incorporate uncertainty into wave-current interaction models, the stochastic elements were introduced into the wave dynamics of two well-established models: the generalized Lagrangian mean model and the Craik–Leibovich model (D.H. Darryl , 2021) [8].

Statement of the problem for three two-layer models

We investigate the problem of internal wave propagation in three hydrodynamic systems: "half-space - half-space", "half-space - layer with rigid lid", and "layer with solid bottom - layer with rigid lid". In each system the low area Ω_1 has density ρ_1 and the upper area Ω_2 has density ρ_2. The areas Ω_1 and Ω_2 are separated by a contact surface $z = \eta(x, y, t)$. The gravitational force is directed perpendicular to the interface surface in the negative z-direction, and the fluids are considered incompressible.

The mathematical statement of the three problems are provided below.

The velocity of wave packet propagation in the areas Ω_1 and Ω_2 is expressed through the gradients of potentials and must satisfy the equation:

$$\varphi_{1,xx} + \varphi_{1,yy} + \varphi_{1,zz} = 0, \quad (1)$$

$$\varphi_{2,xx} + \varphi_{2,yy} + \varphi_{2,zz} = 0, \quad (2)$$

kinematic conditions at the interface surface $z = \eta(x, y, t)$:

$$\eta_{,t} + \alpha\varphi_{1,x}\eta_{,x} + \alpha\varphi_{1,y}\eta_{,y} = \varphi_{1,z}, \quad (3)$$

$$\eta_{,t} + \alpha\varphi_{2,x}\eta_{,x} + \alpha\varphi_{2,y}\eta_{,y} = \varphi_{2,z}, \quad (4)$$

dynamic condition at the interface surface $z = \eta(x, y, t)$:

$$\varphi_{1,t} - \rho\varphi_{2,t} + (1-\rho)\eta + \frac{\alpha}{2}(\nabla\varphi_1)^2 + \frac{\alpha}{2}\rho(\nabla\varphi_2)^2 - (\eta_{,xx} + \eta_{,yy}) = 0, \quad (5)$$

the boundary conditions for the systems take the form:

"half-space - half-space"

$$\varphi_{1,z} \to 0 \quad \text{as} \quad z \to -\infty, \quad (6)$$

$$\varphi_{2,z} \to 0 \quad \text{as} \quad z \to +\infty, \quad (7)$$

"half-space - layer with rigid lid"

$$\varphi_{1,z} \to 0 \quad \text{as} \quad z \to -\infty, \quad (8)$$

$$\varphi_{2,z} = 0 \quad \text{as} \quad z = h, \quad (9)$$

"layer with solid bottom - layer with rigid lid"

$$\varphi_{1,z} \to 0 \quad \text{as} \quad z = -h_1, \quad (10)$$

$$\varphi_{2,z} = 0 \quad \text{as} \quad z = h_2, \quad (11)$$

where φ_j is the velocity potential in Ω_j, η is the interface surface elevation, ρ is the ratio of the density of the upper area to the density of the lower area, h is the thickness of the upper layer in the system "half-space - layer with rigid lid" system, h_1 and h_2 are the depths of the lower and upper layers, respectively, in the system "layer with solid bottom - layer with rigid lid". It's important to note that this problem formulation uses dimensionless variables. Additionally, the parameter α is a non-dimensional parameter introduced to account for the contribution of nonlinear terms in subsequent transformations.

The process of the problems solving

The solutions to the three problems (1) - (5), each of the problems with different boundary conditions, will be sought in the form of Fourier-Stieltjes integrals:

"half-space - half-space" with boundary conditions (6) and (7):

$$\varphi_1 = \int A(\mathbf{q}) \exp(i\theta + kz) d\mathbf{q},$$
$$\varphi_2 = \int B(\mathbf{q}) \exp(i\theta - kz) d\mathbf{q}, \tag{12}$$
$$\eta = \int C(\mathbf{q}) \exp(i\theta) d\mathbf{q};$$

"half-space - layer with rigid lid" with boundary conditions (8) and (9):

$$\varphi_1 = \int A(\mathbf{q}) \exp(i\theta + kz) d\mathbf{q},$$
$$\varphi_2 = \int B(\mathbf{q}) \exp(i\theta)(\exp(k(z-h)) + \exp(-k(z-h)) d\mathbf{q} \tag{13}$$
$$\eta = \int C(\mathbf{q}) \exp(i\theta) dq;$$

"layer with solid bottom - layer with rigid lid" with boundary conditions (10) and (11):

$$\varphi_1 = \int A(\mathbf{q}) \exp(i\theta)(\exp(k(z+h_1)) + \exp(-k(z+h_1))) d\mathbf{q}, \tag{14}$$
$$\varphi_2 = \int B(\mathbf{q}) \exp(i\theta)(\exp(k(z-h_2)) + \exp(-k(z-h_2))) d\mathbf{q},$$
$$\eta = \int C(\mathbf{q}) \exp(i\theta) d\mathbf{q}.$$

where $\mathbf{k} = (k_x, k_y)$ is a wave vector, $k = |\mathbf{k}|$, $\theta = k_x x + k_y y - t\omega$ is a phase, $A(\mathbf{q})$, $B(\mathbf{q})$, $C(\mathbf{q})$ are the random amplitudes of the respective fields that depend on $\mathbf{q} = (\mathbf{k}, \omega)$. Integration in (12) - (14) and subsequent formulas is carried out within the set of real numbers $(-\infty; +\infty)$. In the following steps, our task is to use (12) - (14) and the formulations (1) - (11) to derive equations for the random amplitude $C(\mathbf{q})$.

Using the methodology described in (A. Masuda et all, 1979) [9], we obtain a dynamic equation for the random amplitude $C(\mathbf{q})$ for each of the models in the form

$$\begin{aligned} &-W(\mathbf{q})C(\mathbf{q}) = \\ &= \alpha \int f_2(\mathbf{q},\mathbf{q}_1) C(\mathbf{q}-\mathbf{q}_1) C(\mathbf{q}_1) d\mathbf{q}_1 + \\ &+ \alpha^2 \iint f_3(\mathbf{q},\mathbf{q}_1,\mathbf{q}_2) C(\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2) C(\mathbf{q}_1) C(\mathbf{q}_2) d\mathbf{q}_1 d\mathbf{q}_2, \end{aligned} \tag{15}$$

where the functions $W(\mathbf{q})$, $f_2(\mathbf{q},\mathbf{q}_1)$ and $f_3(\mathbf{q},\mathbf{q}_1,\mathbf{q}_2)$ have the following forms for the different models:

"half-space - half-space"

$$\begin{aligned} &W(\mathbf{q}) = -\frac{\omega^2}{k}(1+\rho) + (1-\rho+k^2), \\ &f_2(\mathbf{q},\mathbf{q}_1) = \frac{(1-\rho)}{2}\Big[\omega(\omega-\omega_1) <\mathbf{k},\mathbf{k}-\mathbf{k}_1> + \\ &+ \omega\omega_1 <\mathbf{k},\mathbf{k}_1> + \omega_1(\omega_1-\omega) <\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1> - \\ &- \omega^2 - \omega_1^2 + \omega\omega_1\Big], \\ &f_3(\mathbf{q},\mathbf{q}_1,\mathbf{q}_2) = \frac{(1+\rho)\omega\omega_2}{2k}\Big[\frac{(\mathbf{k}_2,\mathbf{k}_2+2\mathbf{k}_1)}{2} - \\ &- (\mathbf{k},\mathbf{k}-\mathbf{k}_1) <\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1>\Big] + \\ &+ \frac{(1+\rho)\omega\omega_1}{2k}\Big[\frac{(\mathbf{k}_1,\mathbf{k}_1+2\mathbf{k}_2)}{2} - \\ &- (\mathbf{k},\mathbf{k}-\mathbf{k}_2) <\mathbf{k}_1,\mathbf{k}-\mathbf{k}_2>\Big] + \\ &+ \frac{(1+\rho)}{2}\Big[\frac{(\omega-\omega_1)\omega_2}{k_2}(\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1) + \\ &+ \frac{(\omega-\omega_2)\omega_1}{k_1}(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_2)\Big] + \frac{(1+\rho)\omega_1\omega_2}{2} \times \\ &\times \Big[(1- <\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1>)\frac{(\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1)}{k_2} + \\ &+ (1- <\mathbf{k}_2,\mathbf{k}-\mathbf{k}_2>)\frac{(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_2)}{k_1}\Big] - \\ &- \frac{(1+\rho)\omega_1\omega_2}{2}(1- <\mathbf{k}_1,\mathbf{k}_2>)(k_1+k_2) - \\ &- \frac{(1+\rho)}{2}(\omega_1^2 k_1 + \omega_2^2 k_2); \end{aligned} \tag{16}$$

"half-space - layer with rigid lid"

$$
\begin{aligned}
&W(\mathbf{q}) = -\frac{\omega^2}{k}(1+\rho\,\mathrm{cth}(kh)) + (1-\rho+k^2),\\
&f_2(\mathbf{q},\mathbf{q}_1) = \frac{(1-\rho)}{2}\times\\
&\times\Big[\omega(\omega-\omega_1) < \mathbf{k},\mathbf{k}-\mathbf{k}_1 > \mathrm{cth}(kh)+\\
&+\omega_1(\omega_1-\omega) < \mathbf{k}_1,\mathbf{k}-\mathbf{k}_1 > \mathrm{cth}(k_1h)-\\
&-\omega^2-\omega_1^2+\omega\omega_1+\omega\omega_1 < \mathbf{k},\mathbf{k}_1 > \Big],\\
&f_3(\mathbf{q},\mathbf{q}_1,\mathbf{q}_2) = \frac{(1+\rho\,\mathrm{cth}(kh))}{2}\times \qquad (17)\\
&\times\Bigg(\frac{\omega\omega_2}{k}\Big[\frac{(\mathbf{k}_2,\mathbf{k}_2+2\mathbf{k}_1)}{2}-\\
&-(\mathbf{k},\mathbf{k}-\mathbf{k}_1) < \mathbf{k}_2,\mathbf{k}-\mathbf{k}_1 > \Big]+\\
&+\frac{\omega\omega_1}{k}\Big[\frac{(\mathbf{k}_1,\mathbf{k}_1+2\mathbf{k}_2)}{2}-\\
&-(\mathbf{k},\mathbf{k}-\mathbf{k}_2) < \mathbf{k}_1,\mathbf{k}-\mathbf{k}_2 > \Big]+\\
&+\Big[\frac{(\omega-\omega_1)\omega_2}{k_2}(\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1)\,\mathrm{cth}(k_1h)+\\
&+\frac{(\omega-\omega_2)\omega_1}{k_1}(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_2)\,\mathrm{cth}(k_2h)\Big]+\\
&+\omega_1\omega_2\Big[(1- < \mathbf{k}_1,\mathbf{k}-\mathbf{k}_1 >)\times\\
&\times\frac{(\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1)}{k_2}\,\mathrm{cth}(k_1h)+\\
&+(1- < \mathbf{k}_2,\mathbf{k}-\mathbf{k}_2 >)\frac{(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_2)}{k_1}\,\mathrm{cth}(k_2h)\Big]-\\
&-\omega_1\omega_2(1- < \mathbf{k}_1,\mathbf{k}_2 >)\Big[k_1\,\mathrm{cth}(k_1h)+\\
&+k_2\,\mathrm{cth}(k_2h)\Big]-(\omega_1^2k_1+\omega_2^2k_2)\,\mathrm{cth}(kh)\Bigg);
\end{aligned}
$$

"layer with solid bottom - layer with rigid lid"

$$
\begin{aligned}
&W(\mathbf{q}) = -\frac{\omega^2}{k}(\mathrm{cth}(kh_1)+\rho\,\mathrm{cth}(kh_2))+\\
&+(1-\rho+k^2),\\
&f_2(\mathbf{q},\mathbf{q}_1) = \frac{(1-\rho)}{2}\times \qquad (18)\\
&\times\Big[\omega(\omega-\omega_1) < \mathbf{k},\mathbf{k}-\mathbf{k}_1 > \times\\
&\times\,\mathrm{cth}(kh_1)\,\mathrm{cth}(k_1h_2)+\omega\omega_1 < \mathbf{k},\mathbf{k}_1 > +\\
&+\omega_1(\omega_1-\omega) < \mathbf{k}_1,\mathbf{k}-\mathbf{k}_1 > \times\\
&\times\,\mathrm{cth}(k_1h_1)\,\mathrm{cth}(kh_2-\omega^2-\omega_1^2+\omega\omega_1)\Big],\\
&f_3(\mathbf{q},\mathbf{q}_1,\mathbf{q}_2) = \frac{\mathrm{cth}(kh_1)+\rho\,\mathrm{cth}(kh_2)}{2}\times\\
&\Bigg(\frac{\omega\omega_2}{k}\Big[\frac{(\mathbf{k}_2,\mathbf{k}_2+2\mathbf{k}_1)}{2}-\\
&-(\mathbf{k},\mathbf{k}-\mathbf{k}_1) < \mathbf{k}_2,\mathbf{k}-\mathbf{k}_1 > \Big]+\\
&+\frac{\omega\omega_1}{k}\Big[\frac{(\mathbf{k}_1,\mathbf{k}_1+2\mathbf{k}_2)}{2}-\\
&-(\mathbf{k},\mathbf{k}-\mathbf{k}_2) < \mathbf{k}_1,\mathbf{k}-\mathbf{k}_2 > \Big]+\\
&\frac{(\omega-\omega_1)\omega_2}{k_2}(\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1)\,\mathrm{cth}(k_1h_1)\,\mathrm{cth}(k_1h_2)+\\
&\frac{(\omega-\omega_2)\omega_1}{k_1}(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_2)\,\mathrm{cth}(k_2h_1)\,\mathrm{cth}(k_2h_2)+\\
&\omega_1\omega_2\Big[(1- < \mathbf{k}_1,\mathbf{k}-\mathbf{k}_1 >)\times\\
&\times\frac{(\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1)}{k_2}\,\mathrm{cth}(k_1h_1)\,\mathrm{cth}(k_1h_2)+\\
&+(1- < \mathbf{k}_2,\mathbf{k}-\mathbf{k}_2 >)\frac{(\mathbf{k}_1,\mathbf{k}-\mathbf{k}_2)}{k_1}\times\\
&\times\,\mathrm{cth}(k_2h_1)\,\mathrm{cth}(k_2h_2)\Big]-\\
&-\omega_1\omega_2(1- < \mathbf{k}_1,\mathbf{k}_2 >)\times\\
&\times\Big[k_1\,\mathrm{cth}(k_1h_1)\,\mathrm{cth}(k_1h_2)+\\
&+k_2\,\mathrm{cth}(k_2h_1)\,\mathrm{cth}(k_2h_2)\Big]-\\
&-(\omega_1^2k_1+\omega_2^2k_2)\,\mathrm{cth}(kh_1)\,\mathrm{cth}(kh_2)\Bigg).
\end{aligned}
$$

In (16) - (18), the operator represents the cosine of the angle between the argument vectors. The functions $f_2(\mathbf{q},\mathbf{q}_1)$ and $f_3(\mathbf{q},\mathbf{q}_1,\mathbf{q}_2)$ satisfy

$$
\begin{aligned}
&f_2(\mathbf{q},\mathbf{q}_1) = f_2(\mathbf{q}_1,\mathbf{q}), \quad f_2(\mathbf{q},0) = 0\\
&f_2(\mathbf{q},\mathbf{q}-\mathbf{q}_1) = f_2(\mathbf{q},\mathbf{q}_1), \qquad (19)\\
&f_2(\mathbf{q},\mathbf{q}_1,\mathbf{q}_2) = f_2(\mathbf{q},\mathbf{q}_2,\mathbf{q}_1),
\end{aligned}
$$

which coincides with the previously obtained result for the study of random surface waves (A. Masuda et all, 1979) [9].

For further analysis of equation (15), we will perform ensemble averaging using the methodology described in (L.J. Tick, 1959) [10]. To do this, we will write the expansion of $C(\mathbf{q})$

$$C(\mathbf{q}) = C_1(\mathbf{q}) + \alpha C_2(\mathbf{q}) + \alpha^2 C_3(\mathbf{q}) + \dots \qquad (20)$$

We will perform the averaging procedure according to the rules

$$
\begin{aligned}
&[C_1(\mathbf{q})C_1(\mathbf{q}_1)] = S_{11}\delta(\mathbf{q}+\mathbf{q}_1),\\
&[C_1(\mathbf{q})C_1(\mathbf{q}_1)C_1(\mathbf{q}_2)] = 0,\\
&[C_1(\mathbf{q})C_1(\mathbf{q}_1)C_1(\mathbf{q}_2)C_1(\mathbf{q}_3)] = \qquad (21)
\end{aligned}
$$

$$= S_{11}(\mathbf{q})S_{11}(\mathbf{q}_2)\delta(\mathbf{q}+\mathbf{q}_1)\delta(\mathbf{q}_2+\mathbf{q}_3)+$$
$$+ S_{11}(\mathbf{q})S_{11}(\mathbf{q}_2)\delta(\mathbf{q}+\mathbf{q}_3)\delta(\mathbf{q}_1+\mathbf{q}_2)+$$
$$+ S_{11}(q)S_{11}(\mathbf{q}_1)\delta(\mathbf{q}+\mathbf{q}_2)\delta(\mathbf{q}_1+\mathbf{q}_3).$$

Substituting (20) into (15), which is multiplied by its conjugate $\bar{C}(\mathbf{q})$, and using (21), we obtain the equation:

$$W(\mathbf{q})S(\mathbf{q}) = \int \frac{G(\mathbf{q},\mathbf{q}_1)}{W(\mathbf{q})} S_{11}(q_1)S_{11}(\mathbf{q}-\mathbf{q}_1)d\mathbf{q}_1 - - S_{11}(\mathbf{q}) \int F(\mathbf{q},\mathbf{q}_1)S_{11}(\mathbf{q}_1)d\mathbf{q}_1, \quad (22)$$

where

$$G(\mathbf{q},\mathbf{q}_1) = 2(f_2(\mathbf{q},\mathbf{q}_1))^2,$$
$$F(\mathbf{q},\mathbf{q}_1) = -\frac{4(f_2(\mathbf{q},\mathbf{q}_1))^2}{W(\mathbf{q}-\mathbf{q}_1)} - - f_3(\mathbf{q},\mathbf{q}_1,-\mathbf{q}_1) - 2f_3(\mathbf{q},\mathbf{q},\mathbf{q}_1),$$

which coincides with the results obtained in (A. Masuda et all, 1979) [9] for the problem of random surface gravity wave motion. In (22), S_{11} represents the spectrum of the first harmonic of random waves. Further, based on (22), equations describing free and trapped random internal gravity waves can be derived.

Comparative analysis of the results

Analysing expressions (16) - (18), it is worth noting that the functions $f_2(\mathbf{q},\mathbf{q}_1)$ and $f_3(\mathbf{q},\mathbf{q}_1,\mathbf{q}_2)$ describe two-wave and three-wave interactions, while the function $W(q)$ represents the contribution of dispersion to wave motion. Therefore, by setting $W(q) = 0$, we obtain dispersion relations for the propagation of internal waves, as presented in the works ([I.T. Selezov et all, 2010) [11] and (O.V. Avramenko et all, 2016) [12].

If we consider the "half-space - layer with rigid lid" model in the limit where $h \to +\infty$, then expressions (17) for $W(\mathbf{q})$, $f_2(\mathbf{q},\mathbf{q}_1)$ and $f_3(\mathbf{q},\mathbf{q}_1,\mathbf{q}_2)$ converge to (16), confirming the validity of the obtained results. For the "layer with solid bottom - layer with rigid lid" model, in the case where $h_1 \to -\infty$, the expressions (18) for two-wave and three-wave interactions, as well as dispersion, transform into the corresponding expressions (17) for the "half-space - layer with rigid lid" model (provided $h = h_2$). However, in the "layer with solid bottom - layer with rigid lid" model, if we consider the limiting case where $h_1 \to -\infty$ and $h_2 \to +\infty$, then for the functions $W(\mathbf{q})$, $f_2(\mathbf{q},\mathbf{q}_1)$ and $f_3(\mathbf{q},\mathbf{q}_1,\mathbf{q}_2)$ we obtain expressions (16). Also, it can be added that in the "half-space to half-space" model, when $\rho = 0$ (effectively having no upper layer), the function $f_2(\mathbf{q},\mathbf{q}_1)$ coincides with the analogous function for the surface wave model on the contact surface of the half-space, as described in (A. Masuda et all, 1979) [9]. Additionally, the expression for the function $f_3(\mathbf{q},\mathbf{q}_1,\mathbf{q}_2)$ differs by the term $-\frac{\omega_1\omega_2}{2}(1- <\mathbf{k}_1,\mathbf{k}_2>)(k_1+k_2)$ (V.I. Turtyryka, O.V. Avramenko, 2021) [13].

Conclusions and further developments

As a result of the research, a dynamic equations for the random amplitude are obtained for the hydrodynamical systems: 'half-space - half-space', 'half-space - layer with rigid lid', and 'layer with solid bottom - layer with rigid lid'. For the three systems under consideration, we derived a general form of the dynamic equation using Fourier-Stieltjes integrals. After averaging over amplitudes, dynamic equations in the form of spectra are derived. Analytical expressions describing two-wave and three-wave interactions are obtained. Specific characteristics of the sub-integral functions are identified for each case. An analysis of the obtained expressions is conducted. Further, numerical and analytical investigations of the obtained dynamic equations are planned.

In the subsequent investigations, hydrodynamic two-layer systems with a free surface and three-layer systems bounded by a rigid lid will be examined. It is anticipated that the presence of the third layer and a free surface will lead to the introduction of new terms in the dynamic equation. Furthermore, it will enable the investigation of the interaction between internal and surface waves, characterized in terms of two- and three-wave interactions.

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СЛАБКОНЕЛІНІЙНІ МОДЕЛІ ПОШИРЕННЯ ХВИЛЬ У ДВОШАРОВИХ ГІДРОДИНАМІЧНИХ СИСТЕМАХ

У роботі розглянуто тривимірні моделі поширення стохастичних внутрішніх хвиль у гідродинамічних системах «півпростір – півпростір», «півпростір – шар з кришкою», «шар з твердим дном – шар з кришкою». При побудові моделей шари вважаються ідеальними рідинами, розділеними поверхнею контакту. Основна мета моделювання – отримати динамічне рівняння відносно стохастичної амплітуди поверхневих хвиль. Постановки задач для вказаних моделей наведено в безрозмірному вигляді. Для контролю внеску нелінійних доданків введено безрозмірний нечисловий параметр α. Математична постановка задачі для вказаних моделей містить рівняння руху, кінематичну та динамічну умови на поверхні контакту, умови затухання на нескінченності та умову непротікання на дні та кришці. Для різних моделей відрізняються граничні умови, які визначають загальний вид розв'язків. Розв'язання проводиться в термінах інтегралів Фурье–Стилтьеса. Отримано динамічне рівняння відносно стохастичної амплітуди внутрішніх хвиль

$$-W(\boldsymbol{q})C(\boldsymbol{q}) = \alpha\int f_2(\boldsymbol{q},\boldsymbol{q}_1)C(\boldsymbol{q}-\boldsymbol{q}_1)C(\boldsymbol{q}_1)d\boldsymbol{q}_1 + \alpha^2\iint f_3(\boldsymbol{q},\boldsymbol{q}_1,\boldsymbol{q}_2)C(\boldsymbol{q}-\boldsymbol{q}_1-\boldsymbol{q}_2)C(\boldsymbol{q}_1)C(\boldsymbol{q}_2)d\boldsymbol{q}_1 d\boldsymbol{q}_2,$$

де $\boldsymbol{k}=(k_x,k_y)$ – хвильовий вектор, $k=|\boldsymbol{k}|$, $\theta = k_x x + k_y y - t\omega$ – фаза , $A(\boldsymbol{q})$, $B(\boldsymbol{q})$, $C(\boldsymbol{q})$ - стохастичні амплітуди відповідних полів, які залежать від $\boldsymbol{q}=(\boldsymbol{k},\omega)$. Функції $W(\boldsymbol{q})$, $f_2(\boldsymbol{q},\boldsymbol{q}_1)$ та $f_3(\boldsymbol{q},\boldsymbol{q}_1,\boldsymbol{q}_2)$ отримані для кожної з трьох моделей. Після усереднення по ансамблю амплітуд динамічне рівняння набуває вигляду

$$W(\boldsymbol{q})S(\boldsymbol{q}) = \int \frac{G(\boldsymbol{q},\boldsymbol{q}_1)}{W(\boldsymbol{q})} S_{11}(\boldsymbol{q}_1)S_{11}(\boldsymbol{q}-\boldsymbol{q}_1)d\boldsymbol{q}_1 - S_{11}(\boldsymbol{q})\int F(\boldsymbol{q},\boldsymbol{q}_1)S_{11}(\boldsymbol{q}_1)d\boldsymbol{q}_1,$$

де $G(\boldsymbol{q},\boldsymbol{q}_1) = 2(f_2(\boldsymbol{q},\boldsymbol{q}_1))^2$, $\quad F(\boldsymbol{q},\boldsymbol{q}_1) = -\frac{4(f_2(\boldsymbol{q},\boldsymbol{q}_1))^2}{W(\boldsymbol{q}-\boldsymbol{q}_1)} - f_3(\boldsymbol{q},\boldsymbol{q}_1,-\boldsymbol{q}_1) - 2f_3(\boldsymbol{q},\boldsymbol{q},\boldsymbol{q}_1)$,

Функції $f_2(\boldsymbol{q},\boldsymbol{q}_1)$ та $f_3(\boldsymbol{q},\boldsymbol{q}_1,\boldsymbol{q}_2)$ описують дво- і трихвильові взаємодії у гідродинамічному середовищі. Виявлено внесок дисперсії у хвильовий рух. Розглянуто граничні випадки для досліджуваних моделей, в яких вони переходять одна в одну. Зокрема, в моделі «півпростір – півпростір» при прямуванні щільності верхнього шару до нуля (фактично за відсутності верхнього шару) двохвильові взаємодії якісно збігаються з випадком моделі поверхневих хвиль на поверхні контакту півпростору. При цьому для трихвильових взаємодій виявлено новий доданок.

Ключові слова: стохастичні хвилі, внутрішні хвилі, моделі поширення хвиль.

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