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FRACTIONAL CALCULUS AND ITS APPLICATION IN FINANCIAL MATHEMATICS

Fractional calculus extends classical calculus by allowing differentiation and integration of non-integer orders, providing valuable tools for analyzing complex systems. In this part of the paper we demonstrate the main methods of fractional calculus, including Euler's, Riemann-Liouville, and Caputo approaches. The behavior of functions such as x^n, $e^{\lambda x}$, and $\sin(x)$ is analyzed for fractional orders, demonstrating how fractional differentiation results in varying patterns of growth and decay.

The second part explores the application of fractal derivatives in financial mathematics. We present the use of the Riemann-Liouville derivative to model stock prices in illiquid markets, where the price of an asset may remain unchanged for some time. For this, subdiffusion processes and a fractal integro-differential equation with the Riemann-Liouville derivative are used. The idea of subdiffusion models is to replace the calendar time t in the risk-free bond motion and classical GBM by some stochastic process H_t, which represents a hitting time, which is interpreted as the first time at which G_t hits the barrier t.

Next, we focus on the pricing of a European option when the underlying asset is illiquid. The option price is found as a solution to a fractal Dupire integro-differential equation, in which the time derivative is replaced by the Dzerbayshan– Caputo (D–K) derivative. The D–K derivative is a generalization of the Caputo approach. The form of the D–K derivative depends on a random process G_t, called the subordinate. We take a standard inverse Gaussian process with parameters (1,1) as the subordinate G_t and formulate the Proposition about the form of the fractal Dupire equation for the chosen subordinate. These approaches provide tools that allow the investor to take into account the illiquidity of the financial markets.

Keywords: fractional calculus, Riemann-Liouville derivative, Euler's approach, Riemann-Liouville approach, Caputo's approach, subdiffusion, Dupire equation, Black-Scholes model, Partial Integro-Differential Equations, Dzerbayshan–Caputo derivatives, subordinator.

Introduction

Differentiation and integration are fundamental concepts in mathematics that have been studied intensively for centuries. In its simplest form, differentiation involves calculating the slope of a function at a given point, while integration involves finding the area under a curve. These concepts are well known and have been thoroughly studied over the years, leading to clear and well-known results that are widely used in a wide variety of fields.

An interesting question is the existence of differentiation and integration for fractional order, the so-called fractional calculus. As explained in [2], the classical derivative restricted by rate of change falls short to describe many phenomena that could not be constructed properly by integer order calculus encompassed by fractional calculus. Due to this fact, fractional derivatives are proposed for capturing the past history as in the classical integration. Hence, both fractional derivative and integral have past memory making them much more advantageous than classical counterparts. The history of fractional calculus can be traced back to the work of Euler and Laplace in the 18th century. Later, other prominent mathematicians such as Caputo, Liouville, and Riemann also made significant contributions to the field. Over the past few decades, this branch of mathematical analysis has gained attention due to its significant potential for applications in various fields including physics, engineering, finance, and biology. The main idea of fractional calculus is to extend the concepts of differentiation and integration to functions with non-integer orders. This allows for a more accurate description of complex phenomena, such as anomalous diffusion [7], viscoelasticity [11], and fractal behaviour [12].

The purpose of this paper is to study approaches to fractional calculus, illustrate them by visualizing the results in the Python programming language and demonstrate how Dzerbayshan– Caputo (D–C) derivative is used for option evaluating. By achieving this goal, this study aims to fill the gap in the existing literature on this topic and provide a better understanding of the potential of fractional calculus in financial mathematics.

The paper is organized as follows. The second section consists of two subsections. In the first subsection the comparison between classical and fractional calculus interpretations is discussed. Also we review the main approaches to fractional calculus: Euler, Liouville, Riemann, and Caputo. The second subsection focuses on the Riemann–Liouville approach to fractional calculus. This approach builds upon the Riemann method and the Cauchy integral formula, allowing for the generalization of integration to non-integer orders using the Gamma function. The fractional integral is defined, and its important properties, such as the additive property of fractional integrals and the relationship between fractional integration and differentiation, are discussed. The Riemann-Liouville approach has a huge application in financial mathematics and it is used for stock price modeling on illiquid markets. The Caputo's approach modifies the Riemann-Liouville definition to simplify initial condition handling in fractional differential equations, making it highly valuable for real-world modeling. The updated approach is known as Dzerbayshan–Caputo derivative introduced later and is applied for option pricing on illiquid markets. In the last subsection, we examine how the fractional order α influences the behavior of derivatives across the Euler, Caputo, and Riemann-Liouville approaches. The behavior of functions such as x^n, $e^{\lambda x}$, and $\sin(x)$ is analyzed for fractional orders, demonstrating how fractional differentiation results in varying patterns of growth and decay.

The third section is devoted to the applications of fractional calculus in financial mathematics, particularly for describing the dynamics of the illiquid markets. Classical models, like Black-Scholes, assumes that asset prices follow Brownian motion, a process with independent and stationary increments. However, these models often fail to account for the irregularities and memory effects observed in illiquid markets, where asset prices exhibit anomalous behaviors like stationarity or jumps. In this context, fractional calculus and subdifusive models which incorporate hitting times and irregular trading activity provides a natural extension to incorporate such complexities, offering a more accurate representation of the underlying dynamics of financial illiquid assets.

First, we mention the usual model of subdiffusion, which is the celebrated Fractional Fokker-Planck equation (see for example [8]). This equation is based on the Riemann- Liouville fractional derivative and describes the probability density function $w(t)$ of the sub-diffusive stock process. This theory fully detailed in the literature (see for example [7], [6], [8]). The application of the Fractional Fokker-Planck equation to the risk measuring in financial mathematics you can find in [22].

After that we focus on the option pricing problem under subdiffusion. The main idea of subdiffusive is to replace calendar time t by hitting time H_t, which interpreted as the first time at which stochastic process (so called subordinator) G_t hits the barrier t. Initially for the option pricing under subdiffusion was used the method of discounted mathematical expectation of the payoff function under risk-neutral measure (see for example [7], [6], [23], [24]). A new method was proposed recently by the Donaten and Leonenko (see [20]), which uses a fractional Dupire equation with Dzerbayshan–Caputo derivatives for deriving the European call option.

In this study we just apply the idea of this approach for the standard IG process (SIG), which simplifies the equation and recovers the fractional Dupire form under specific conditions. It is noteworthy that this approach was detailed for inverse $\alpha-$ stable and inverted Poisson processes in [20], for inverse inversian Gaussian in [21], for Gamma in [22].

Finally, we formulate the proposition about application of the fractal Dupire PIDE in the case of the SIG subordinator. By incorporating fractional calculus, we have used for SIG a model that captures the non-local and memory-dependent nature of market dynamics, offering a more accurate and flexible tool for pricing financial instruments in such environments.

Interpretations and approaches to fractional calculus

Main approaches to fractional calculus. Fractional calculus is an extension of traditional integral integration and differentiation. Similarly, fractional exponents are an extension of integer exponents.

Integer calculus has clear and well-known physical and geometric interpretations. For example, the geometric value of a first-order derivative at some point x_0 is equal to the tangent of the tangent line to the graph of the function at the point with the abscissa x_0 and is equal to the angular coefficient of this tangent line.

In the case of differentiation and integration of arbitrary order, there were no clear geometric and physical interpretations for almost 300 years. Eventually, however, interpretations were found. In [10], the geometric interpretation is the so-called 'shadows on the walls', and the physical interpretation is 'shadows of the past'.

Here is an explanation of what exactly these interpretations are. For example, the geometric interpretation of fractional integration is to add a third dimension to the standard pair $\tau, f(\tau)$. If τ is time, then the added dimension can be called a 'deformed' timescale. The physical or mechanical interpretation of fractional calculus is to use two types of time in calculations: cosmic and individual.

Since this paper is devoted more to the mathematical side of the issue, it is worth describing the 'shadows on the walls' in a little more detail. The geometric interpretation of the fractional integral is to display the so-called 'fence' on two walls, as is clear from this sentence, fractional calculus provides a third dimension for analysing a function. Together with the 'fence', whose shape changes according to the change of time t from 0 to b, its shades on the walls also change, representing the right-handed Riemann-Liouville fractional integral and the classical integral with a moving lower bound. [10]

The history of fractional calculus starts from the work of Euler and Laplace in the 18th century. In 1730, Euler proposed a generalization of this formula:

$$\frac{(d^n x^m)}{(dx^n)} = m(m-1)\dots(m-n+1)x^{(m-n)}$$

Using the properties of the Gamma function:

$$\Gamma(m+1) = m(m-1)\dots(m-n+1)\Gamma(m-n+1)$$

he came up with the following formula:

$$\frac{(d^n x^m)}{(dx^n)} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{(m-n)}$$

This formula is very useful and easy to use for calculating fractional differentials of functions of the form $f(x) = x^a$, where $a \in R$. [9]

In the period from 1832 to 1855, Liouville proposed three important definitions for fractional calculus. In the first definition, using the exponential representation of the function $f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}$, he generalized $\frac{(d^m e^a x)}{(dx^n)} = a^m e^a x$ as:

$$\frac{d^v f(x)}{dx^v} = \sum_{n=0}^{\infty} c_n a_n^v e^{a_n x}$$

Its second definition is a fractional integral [9]:

$$\int^{\mu} \Phi(\mathrm{x})\mathrm{dx}^{\mu} = \frac{1}{(-1)^{\mu}\Gamma(\mu)}\int_0^{\infty} \Phi(x+\alpha)\alpha^{\mu-1}d\alpha$$

$$\int^{\mu} \Phi(\mathrm{x})\mathrm{dx}^{\mu} = \frac{1}{\Gamma(\mu)}\int_0^{\infty} \Phi(x-\alpha)\alpha^{\mu-1}d\alpha$$

By replacing $x+\alpha$ and $x-\alpha$ with τ in the above formulas, the following formulas were obtained:

$$\int^{\mu} \Phi(\mathrm{x})\mathrm{dx}^{\mu} = \frac{1}{(-1)^{\mu}\Gamma(\mu)}\int_x^{\infty} \Phi(\tau)(\tau-x)^{\mu-1}d\tau$$

$$\int^{\mu} \Phi(\mathrm{x})\mathrm{dx}^{\mu} = \frac{1}{\Gamma(\mu)}\int_x^{\infty} \Phi(\tau)(\tau-x)^{\mu-1}d\tau$$

The third definition is a fractional differential:

$$\frac{d^{\mu}F(x)}{dx^{\mu}} = \frac{(-1)^{\mu}}{h^{\mu}}\Big(F(x)\frac{\mu}{1}F(x+h)+ \\ +\frac{\mu(\mu-1)}{1\cdot 2}F(x+2h)-\cdots$$

$$\frac{d^{\mu}F(x)}{dx^{\mu}} = \frac{1^{\mu}}{h^{\mu}}\Big(F(x)\frac{\mu}{1}F(x-h)+ \\ +\frac{\mu(\mu-1)}{1\cdot 2}F(x-2h)-\cdots$$

From 1847 to 1876, Riemann proposed the other definition of the fractional integral:

$$D^{-v}f(x) = \frac{1}{\Gamma(v)}\int_c^x (x-t)^{v-1}f(t)dt + \psi(t)$$

The Riemann-Liouville definition is one of the two most famous in the field of fractional calculus, it is a combination of the previous two definitions: the definition of the derivative of the Cauchy integral formula and the Riemann definition.

$${}_aD_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}$$

In this formula, n is the so-called 'ceiling' of α, which means that n is the smallest integer greater than the number whose ceiling it is, in our case $n-1 \le \alpha < n$.[9]

Another well-known definition is Caputo's definition, created in 1967, and as mentioned earlier, it is an improvement of the Riemann-Liouville definition for the calculation of fractal equations. [9]

$${}_a^CD_t^{\alpha}f(t) = \frac{1}{\Gamma(\alpha-n)}\int_{\alpha}^t \frac{f^n(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \\ (n-1 \le \alpha < n) \qquad (1)$$

The Riemann–Liouville approach. The Riemann–Liouville approach is based on the Riemann approach and the Cauchy integral formula.

By using the Cauchy formula for repeated integration over parameters, we can calculate the antiderivative α of the function order several times, which leads to the following formula:

$$I^{\alpha} f(t) = \frac{1}{(\alpha-1)!} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

As mentioned in another section, the generalisation of the factorial is the so-called Gamma function. So, to improve the already obtained formula, we will replace this factorial with the Gamma function, generalising the result.

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \alpha > 0$$

This formula is a working formula for fractional integration. It is called the Riemann-Liouville left-handed integral. This integral is considered one of the easiest formulas to understand in the world of fractional calculus. The main note is that α can be a complex number due to the limitations of the Gamma function, but always with a real part greater than zero.

This integral has the following important dependencies:

$$I^{\alpha}\left(I^{\beta} f\right) = I^{\alpha+\beta} f \frac{d}{dx} I^{\alpha+1} f = I^{\alpha} f$$

Unfortunately, we cannot simply say that a differential of order α will be equal to an integral of order $-\alpha$. Due to the presence of the Gamma function in the Riemann-Liouville left-handed integral formula, the use of negative order is not possible, and hence it cannot be used to define a fractional order differential.

To start converting an integral to a differential, you should start with the fact that after differentiating n times, the integration will be equal to the original function itself.

$$\frac{d^n}{dt^n}\left(I^n f(t)\right) = f(t)$$

This means that the derivative is the left-hand side of the integral. However, the integral is not the left-hand side of the derivative because the integral adds an arbitrary constant. That is, in general, the inverse of the previous property is not true. Under this condition, we would still like to be able to define differentiation through operations that are understandable and possible. Such an operation, which has the desired properties, would be:

$$D^{\alpha} f = \frac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} \left(I^{\lceil \alpha \rceil - \alpha} f\right)$$

Here, $\lceil \alpha \rceil$ is the 'ceiling' of α, the result of rounding the number to the next smallest integer greater than the given number. Let's write this record in more detail:

$$_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\, d\tau}{(t-\tau)^{\alpha-n+1}}, \tag{2}$$

where n is the ceiling of α. This is the left-handed Riemann-Liouville fractional derivative. Most fractional calculations are long and complicated, if not completely intractable, if performed manually without the help of a computer.

Illustration of fractional calculus approaches to some basic functions. In this subsection, we will illustrate and visualize the Euler, Riemann-Liouville, and Caputo approaches to fractional calculus for some functions.

a) Euler's approach.

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \tag{3}$$

The simplest example is the following function:

$$f(x) = 1$$

for which:

$$\frac{d^{\alpha} 1}{dx^{\alpha}} = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$$

In this case, we substitute $m = 0, n = \alpha$, where $alpha$ is the order of differentiation, into the formula (3). Using Python, we visualize the graphs of the differentials of the function $f(x) = 1$ for the following orders: $\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}$.

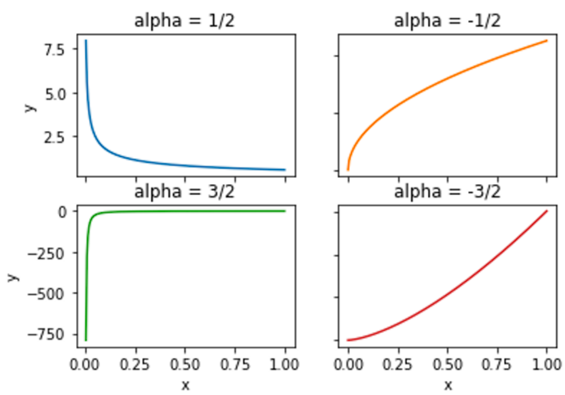

Figure 1. Graphical representation of the fractional differentials of the function $f(x) = 1$ using the formula (3) in Python.

In order to obtain Figure 1, the matplotlib library was used to calculate the result of the Gamma function, using the gamma() method, which returns the value of the function depending on the input x. Another example is solved below for Euler's formula, in this case the function has the following form:

$$f(x) = x$$

Let's repeat the steps described above. In formula (2), we will make the following substitutions: $m = 1$, $n = \alpha$, where α is again the order of differentiation. After performing these steps, we will get the following function:

$$\frac{d^\alpha x}{dx^\alpha} = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)}$$

Again using Python and its matplotlib library and the gamma() method, we will calculate and display the graphs of the differentials of the following function of orders: $\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{3}{2}$.

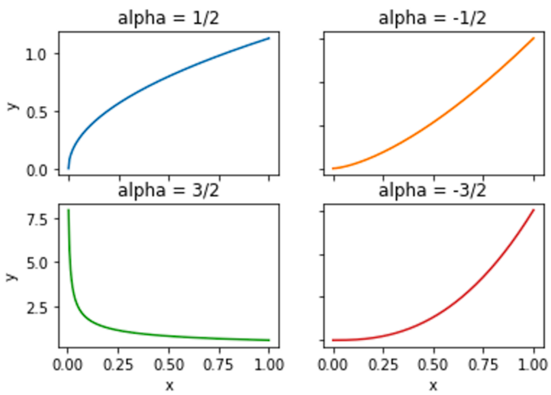

Figure 2. Graphical representation of the fractional differentials of the function $f(x) = x$ using formula (2) in Python.

As you can see from the previous examples, the Euler approach is very convenient for calculating fractional differentials of functions of a type:

$f(x) = x^n$, $n \in Q$. Analyzing the graphs of the derivative functions shown in Figures 1 and 2, we can draw the following conclusion: there is no single law by which these functions are constructed. For example, for $\alpha = 3/2 f(x) = 1$ will be monotonic and strictly increasing, and for $f(x) = x$ the fractional differential will give us a monotonic strictly decreasing function. Similarly, for $\alpha = 1/2$, the function is strictly decreasing for $f(x) = 1$ and strictly increasing for $f(x) = x$. While for the other two alphas, no such dynamics is observed.

b) Caputo's approach

At first glance, this approach (see (1)) seems overly complicated and requires too many calculations. However, according to Theorrm 5 of Maria Ishtev [5], the differential of an exponential function is of the form:

$$f(x) = e^{\lambda x}$$

and after a number of transformations, it looks like:

$$\frac{d^\alpha e^{\lambda x}}{dx^\alpha} = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} x^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = \lambda^n x^{n-\alpha} E_{1,n-\alpha+1}, \quad (4)$$

where $\lambda \in C, n-1 < \alpha < n, \alpha \in R, n \in N$. The proof of this theorem is based on the generalised Mittag-Lefler function for two parameters:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0,\ \alpha, \beta \in R,\ z \in C$$

and the facts from [5]:

$$D_*^\alpha f(t) = D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0), \quad t > 0, \alpha \in R,\ n-1 < \alpha < n$$

and

$$D^\alpha e^{\lambda t} = t^{-\alpha} E_{1,1-\alpha}(\lambda t).$$

With this formula, we can already write solutions for several examples. Let's start with the function:

$$f(x) = e^x$$

Let's use the formula (4) and get it:

$$\frac{d^\alpha e^x}{dx^\alpha} = \sum_{k=0}^{\infty} \frac{x^{k+n-\alpha}}{\Gamma(k+n+1-\alpha)}$$

Using WolframAlpha, we visualize graphs of order differentials: $\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}$.

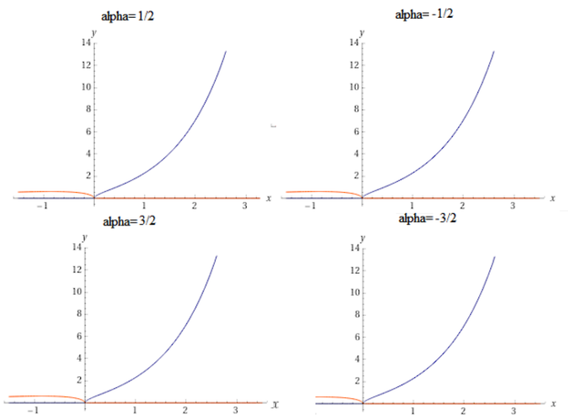

Figure 3. Graphical representation of the fractional differentials of the function $f(x) = e^x$ using the formula (4), using WolframAlpha.

For the next example, let's look at a function:

$$f(x) = e^{2x}$$

Using the formula (4), we get the following differential function:

$$\frac{d^\alpha e^{2x}}{dx^\alpha} = \sum_{k=0}^{\infty} \frac{2^{k+n} x^{k+n-\alpha}}{\Gamma(k+n+1-\alpha)}$$

Using WolframAlpha, we visualise graphs of order differentials: $\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}$.

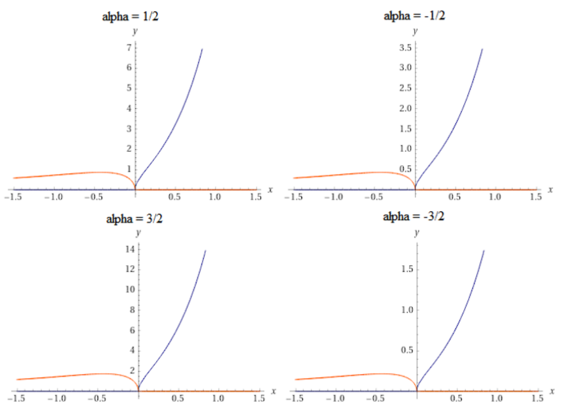

Figure 4. Graphical representation of the fractional differentials of the function $f(x) = e^{2x}$ using the formula (4) in WolframAlpha.

Analyzing the graphs of the derivative functions shown in Figures 3 and 4, we can draw the following conclusion: these functions have a clear pattern. It can be noted that in both figures, the graphs correspond to the behaviour of the integral differential for functions of the form $f(x) = e^{nx}$. Thus, we see that the change in α changes the y-value of the point of intersection of the graphs with the ordinate axis. The growth dynamics of the graphs also has a single pattern that corresponds to the whole number.

c) The Riemann-Liouville approach.

The application of the Riemann-Liouville formula (2) requires numarical methods of calculation, which is a separate complex task. It is also important to note that Python library for it has very large limitations. This library contains methods for calculating two approaches: Riemann-Liouville and Grunwald-Letnikov.

Main Function	Usage
GLpoint	Computes the GL differintegral at a point
GL	Computes the GL differintegral over an entire array of function values using the Fast Fourier Transform
GLI	Computes the improved GL differintegral over an entire array of function values
RLpoint	Computes the RL differintegral at a point
RL	Computes the RL differintegral over an entire array of function values using matrix methods

Figure 5. A set of functions and their functionality of the differint library.

Using this library, let's give an example for a trigonometric function:

$$f(x) = \sin(x)$$

Let's use the RL() function to calculate the order differentials: $\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$. And visualise the results using the mathplotlib library:

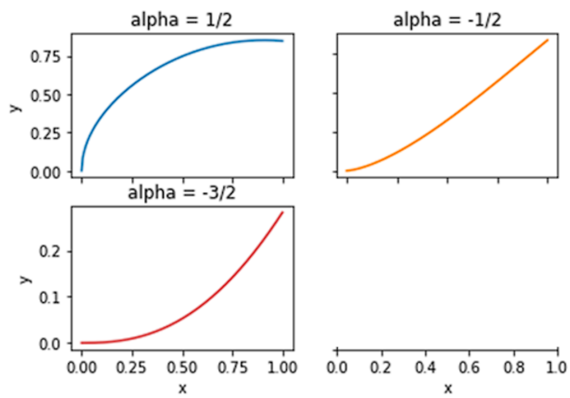

Figure 6. Graphical representation of the fractional differentials of the function $f(x) = \sin(x)$ using the (2) approach, using the Python library differint.

Analyzing the graphs in Figure 6, we can only note the monotonicity of each of the functions shown on it. So for all three values of α used, the functions are increasing. However, we cannot observe any single behaviour that would depend on the order of differentiation and would correspond to the behaviour of the function $f(x) = \sin(x)$ in the integer domain.

Fractional calculus in financial mathematics: Option pricing for subdiffusion model

In traditional financial markets, the Black-Scholes model is widely used for option pricing(see [4], [3]). However, in illiquid markets where trading delays and irregularities occur, classical diffusion models often fall short. Subdiffusive models[1], which incorporate waiting times and irregular trading activity, offer a more accurate way to represent such markets.

The idea of subdiffusion models is to replace the calendar time t in the risk-free bond motion and classical GBM by some stochastic process H_t, which represents a hitting time, defined as:

$$H_t = \inf\{\tau > 0 : G_\tau \geq t\}. \tag{5}$$

and interpreted as the first time at which G_t hits the barrier t. H_t is positive, non-decreasing and has all the properties to be used as a stochastic clock. By construction, the inverted process may be constant. Therefore, any process subordinated by H_t exhibits motionless periods.

The definition (5) of hitting time is based on the use of some other random process called a subordinator G_t. The subordinator G_t is generally a non-decreasing stochastic process.

The usual model of subdiffusion is the celebrated Fractional Fokker-Planck equation (see for example [8]). This equation describes the probability density function $w(t)$ of the sub-diffusive stock process:

$$\frac{\partial w}{\partial t} = {}_aD_t^\alpha \left[-\mu \frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\right] w(x,t), \tag{6}$$

where ${}_aD_t^\alpha f(t)$ is is the left-handed Riemann-Liouville fractional derivative (see sections above and formula (2)). This application of the fractional derivatives to the financial mathematics is a quite important, but the more detailed consideration of this problem is outside the framework of this paper.

This study focuses on the option pricing problem under subdiffusion.

In the classical diffusion model, the fair price of a European call option on an asset with price S_t at time t is provided by the Black-Scholes formula. The alternative way to compute the fair price using Dupire equation [15].

For the subdiffusive we derive the European call price using a fractional Dupire equation with Dzerbayshan–Caputo derivatives[13].

For this aim we start with classical Dupire equation and consider the case when $\sigma(S_t, t) = \sigma$ is the constant. Then the Dupire equation has a form

$$\begin{aligned}\frac{\partial C(T,K)}{\partial T} + r(T)C = \mu(T)C-\\ -\mu(T)K\frac{\partial C}{\partial K} + \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2},\end{aligned} \tag{7}$$

where $C(T,K)$ is the option price at time T with strike price K, $r(T)$ is the risk-free rate, $\mu(T) = r(T) + q(T)$ is the drift, $q(T) = 0$ is the continuous dividend rate and σ is the volatility.

After that we input variable $k = \ln K$. The derivatives with respect to K are then transformed as follows:

$$\frac{\partial}{\partial K} = \frac{1}{K}\frac{\partial}{\partial k}, \quad \frac{\partial^2}{\partial K^2} = \frac{1}{K^2}\frac{\partial^2}{\partial k^2}$$

Substituting these expressions into the original equation, we obtain the Dupire equation in terms of $k = \ln K$:

$$\frac{\partial C(T,k)}{\partial T} = -r\frac{\partial C(T,k)}{\partial k} + \frac{\sigma^2}{2}\frac{\partial^2 C(T,k)}{\partial k^2}. \tag{8}$$

For the option pricing in subdiffusion model we just replace the derivative for the time in the Dupire equation by a Dzerbayshan–Caputo (D–C) derivative (see [20], [13]). So, the fractional Dupire PIDE has a form

$${}^{\Psi}DC_H(T,k) = -r\frac{\partial}{\partial k}C_H(T,k) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial k^2}C_H(T,k),$$

where ${}^{\Psi}Du(t)$ is the convolution-type derivative, called the Dzerbayshan–Caputo (D–C) derivative.

The Dzerbashyan–Caputo derivative generalizes the classical Caputo[14] derivative by incorporating a convolution kernel. This adaptation provides greater flexibility in modeling market behavior influenced by memory effects and irregular temporal dynamics. Specifically, it allows the model to accurately capture the heavy-tailed waiting time distributions and subdiffusive characteristics often observed in illiquid markets. By combining fractional calculus with the dynamics of Lévy subordinators, this approach bridges the gap between theoretical models and observed market anomalies. We focus on this derivative to better account for the stochastic time changes driven by inverse subordinators, making it particularly well-suited for subdiffusive option pricing.

The D–C derivative for a function $u(t)$ is given by:

$$^{\Psi}Du(t) = b\frac{d}{dt}u(t) + \int_0^t \frac{\partial}{\partial t}u(t-s)\nu(s)ds. \quad (9)$$

Here, the function Ψ represents the Lévy exponent associated with the subordinator G_t.

In this study, we use the Standard Inverse Gaussian (SIG) process as the subordinator G_t. Inverse Gaussian (IG) subordinator G_t is a non-decreasing Lévy process, where the increments $G_{t+s} - G_s$ follow the inverse Gaussian distribution $\varrho(\delta t, \gamma)$ with probabilities density function (PDF):

$$g(x,t) = \frac{\delta t}{\sqrt{2\pi x^3}} e^{\delta\gamma t - (\delta^2 t^2/x + \gamma^2 x)/2}, \quad x > 0;$$

and with Lévy measure

$$\tilde{\nu}(dx) = \frac{\delta}{\sqrt{2\pi x^3}} e^{\left(-\frac{\gamma^2 x}{2}\right)} dx, \quad x > 0, t > 0. \quad (10)$$

For $\gamma = \delta = 1$ we have the standard IG distribution in the form

$$f(x,t) = \frac{t}{\sqrt{2\pi x^3}} e^{\left(-\frac{(x-t)^2}{2x}\right)}, \quad x > 0, t > 0.$$

For a given subordinator G_t, its inverse, denoted as H_t, is defined by the hitting time H_t (5). The density function $h(x,t)$ of H_t has an integral representation [16] and for standard IG distribution has a form:

$$h(x,t) = \frac{1}{\pi} e^{x-\frac{1}{2}} \int_0^\infty \frac{e^{-ty}}{y + \frac{1}{2}} (\sin(x\sqrt{2y}) + \\ + \sqrt{2y}\cos(x\sqrt{2y}))dy.$$

The moments of H_t can be numerically evaluated using $h(x,t)$, and explicit formulas for the first and second moments were obtained via Laplace transforms. Asymptotic behavior shows that for large t [17]:

$$E(H_t) \sim \begin{cases} \left(\frac{\gamma}{\delta}\right)t, & \gamma > 0 \\ \left(\frac{1}{\delta}\sqrt{\frac{2t}{\pi}}\right)t, & \gamma = 0, \end{cases}$$

$$Var(H_t) \sim \left(\frac{\gamma}{\delta}\right)^2 t^2.$$

For the standard case ($\delta = 1, \gamma = 1$), we have $E(H_t) \sim t$ and $Var(H_t) \sim t^2$ and this fact explaines why we choose these parameters.

Thus we focus on standard inverse Gausian subordinator(see [18]) G_t Its Lévy-Khintchine representation can be written as:

$$\Psi(x) = \int_0^{+\infty} (1 - e^{-sz})\tilde{\nu}(dz),$$

where $\tilde{\nu}$ is the Lévy measure.

The Lévy measure for standard IG subordinator will be:

$$\tilde{\nu}(dz) = \frac{1}{\sqrt{2\pi z^3}} e^{-\frac{z}{2}} dz, \quad z > 0, , t > 0. \quad (11)$$

Thus, the integral kernel $\nu(s)$ in (11) is the integral of $\tilde{\nu}$ over (s, ∞):

$$\nu(s) = \int_s^{+\infty} \frac{1}{\sqrt{2\pi z^3}} e^{-\frac{z}{2}} dz = \\ = \frac{2e^{-\frac{s}{2}}}{\sqrt{2\pi s}} - \operatorname{erfc}\left(\frac{\sqrt{s}}{\sqrt{2}}\right) = \\ = \frac{2e^{-\frac{s}{2}}}{\sqrt{2\pi s}} + \operatorname{erf}\left(\frac{\sqrt{s}}{\sqrt{2}}\right) - 1$$

Here, $\operatorname{erf}(x)$ denotes the error function, which is related to the standard normal cumulative distribution function $\Phi(x)$:

$$\nu(s) = 2\Phi(s) - 2 + \frac{2e^{-\frac{s}{2}}}{\sqrt{2\pi s}}$$

Then we can represent the D-C derivative as:

$$^{\Psi}Du(t) = 2\int_0^t \frac{\partial}{\partial t}u(t-s)\left(\Phi(s) - 1 + \frac{e^{-\frac{s}{2}}}{\sqrt{2\pi s}}\right)ds.$$

Now, substituting this to (8), we obtain:

$$\int_0^T \frac{\partial}{\partial T}C_H(T-s,k)\left(\Phi(\sqrt{s}) - 1 + \frac{e^{-\frac{s}{2}}}{\sqrt{2\pi s}}\right)ds = \\ = -\frac{r}{2}\frac{\partial}{\partial k}C_H(T,k) + \frac{\sigma^2}{4}\frac{\partial^2}{\partial k^2}C_H(T,k), \quad (12)$$

Thus we can state the following proposition.

Proposition 1. *If the subordinator G_t for the hitting time (5) is the Standard Inverse Gaussian (SIG) process, the fair price $C_H(T,k)$ of the European option with time to maturity T and strike price K is the solution of the PIDE (12), where:*

- *$\Phi(s)$ is the standard normal cumulative distribution function;*
- *r is the risk-free rate;*
- *σ is the asset volatility,*

- $k = \log K$.

It is worth noting, that the Dzherbashyan-Caputo fractional derivative plays a crucial role in this model cause it incorporates the nonlinear dynamics of the market, particularly the delays modeled by the subordinator. The convolution kernel of this derivative includes the function $\Phi(s)$, which captures heavy tails and the slow decay of the waiting time distribution. This enables the model to accurately reflect the behavior of illiquid markets and pricing anomalies.

Remark 1. To solve the PIDE numerically, the time T and space k variables are discretized into grids with steps Δt and Δk (see [20]). The integral term is approximated using the trapezoidal rule or quadrature, while the derivatives $\frac{\partial}{\partial k}$ and $\frac{\partial^2}{\partial k^2}$ are computed with finite difference methods. An implicit time-stepping scheme is used for stability, with initial conditions $C_H(0, k) = (e^k - K)^+$ + and asymptotic boundary conditions applied at $k \to \pm\infty$. The equation is transformed into a system of algebraic equations and solved iteratively using numerical tools like Python or MATLAB, ensuring accuracy and stability of the solution. Another calculation algorithm was presented by Omid Nikan et al [19].

Conclusion

Fractional calculus is a branch of mathematics that extends classical calculus to allow differentiation and integration of non-integer orders. This study looks at the main methods of fractional calculus: Euler's, Riemann-Liouville, and Caputo approaches.

The study analyzes the functions x^n, $e^{\lambda x}$, and $\sin(x)$ for fractional orders like $1/2$, $-1/2$, $3/2$, and $-3/2$. Euler's method was used to find analytical solutions for the functions $f(x) = 1$ and $f(x) = x$. The Riemann-Liouville method was applied to the function $f(x) = \sin(x)$ using the Python `differint` library. The graphs of the differentials showed that the behavior of the functions changes depending on the specific case, with no general pattern across all cases. The graphs demonstrated consistent trends of either growth or decay, similar to what is observed in integer-order differentiation, where different orders of differentiation lead to different behaviors.

The Caputo method was used on the functions $f(x) = e^x$ and $f(x) = e^{2x}$, with approximation methods applied. Unlike the previous cases, the fractional differentials for these functions followed a consistent pattern, with the graphs behaving similarly but differing only in the intersection point with the y-axis, depending on the order of differentiation.

In the Riemann-Liouville approach, the study used the `RL()` function from the `differint` library to calculate the fractional derivatives of the function $f(x) = \sin(x)$ for orders $1/2$, $-1/2$, and $-3/2$. The graphs of the resulting fractional derivatives showed that all the functions exhibited monotonic growth. However, there was no clear pattern that could be attributed to the order of differentiation in the same way as integer-order derivatives.

In financial modeling, the Riemann-Liouville fractional derivative has been used to describe subdiffusive processes, improving the Black-Scholes model by accounting for market features that traditional models don't capture, such as irregular trading and delays. By adding subdiffusion with a fractional Partial Integro-Differential Equation (PIDE) using the Dzerbayshan–Caputo derivative, the model better reflects how asset prices move by considering memory effects and subdiffusive behavior.

The study finds that subdiffusive models are more accurate and sensitive, especially in capturing market behaviors like periods of price stability. However, these models are computationally heavy and not suitable for real-time use by most investors. While fractional calculus is a powerful tool for modeling complex systems like fluids and fractals, it requires a lot of computing power and time. While these models are useful for specific tasks, they are not necessary for general financial applications. Therefore, developing simpler approximation methods remains an important area of research. This study shows that fractional calculus can improve financial modeling.

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Зубріцька Д. Є., Щестюк Н. Ю., Случинський Д. Ю.

ФРАКЦІЙНЕ ЧИСЛЕННЯ ТА ЙОГО ЗАСТОСУВАННЯ У ФІНАНСОВІЙ МАТЕМАТИЦІ

Фракційне числення розширює класичне числення, дозволяючи диференціювання та інтегрування довільного (нецілого) порядку, що надає цінні інструменти для аналізу складних систем. У цій частині роботи ми демонструємо основні методи фракційного числення, зокрема підходи Ейлера, Рімана-Ліувілля та Капуто. Аналізується поведінка функцій, таких як x^n, $e^{\lambda x}$ і $\sin(x)$, для фракційних порядків, що демонструє, як фракційне диференціювання призводить до різних закономірностей зростання та згасання.

У другій частині досліджується застосування фрактальних похідних у фінансовій математиці. Ми представляємо використання похідної Рімана-Ліувілля для моделювання динаміки цін акцій на неліквідних ринках, де вартість активу може залишатися незмінною протягом деякого часу. Для цього використовуються субдифузійні процеси та фрактальне інтегро-диференціальне рівняння з похідною Рімана-Ліувілля.

Ідея субдифузійних моделей полягає в тому, щоб замінити календарний час t у русі безризикової облігації та класичному геометричному броунівському русі (GBM) деяким стохастичним процесом H_t, який є моментом досягнення певного рівня. Його можна інтерпретувати як перший момент, коли процес G_t досягає бар'єру t.

Далі ми зосереджуємося на оцінюванні ціни європейського опціону у випадку, коли базовий актив є неліквідним. Ціна опціону визначається як розв'язок фрактального інтегро-диференціального рівняння Дюпіра, в якому похідна за часом замінюється похідною Джербашяна-Капуто (D–K). Похідна D–K є узагальненням підходу Капуто. Форма похідної D–K залежить від випадкового процесу G_t, який називають субординатою. Ми розглядаємо стандартний обернений гаусівський процес із параметрами (1,1) як субординату G_t і формулюємо твердження про вигляд фрактального рівняння Дюпіра для вибраної субординати.

Завдяки запропонованим підходам інвестор отримує інструменти, що дозволяють йому враховувати неліквідність фінансових ринків.

Ключові слова: фракційне числення, похідна Рімана-Ліувілля, підхід Ейлера, підхід Рімана-Ліувілля, підхід Капуто, субдифузія, рівняння Дюпіра, модель Блека-Шоулза, часткові інтегро-диференціальні рівняння, похідні Джербашяна-Капуто, субордината.

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