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PORTFOLIO OPTIMIZATION FOR REAL DATA: APPROACHES AND CHALLENGES

Portfolio optimization continues to be a dynamic field within finance, integrating new theories and technologies to better meet investor needs. As financial markets evolve, so too will the methodologies used to optimize portfolios, making it an area ripe for ongoing research and innovation.

Classical Markowitz approach is based on the mean-variance optimization, which quantifies the trade-off between risk (variance) and return (expected return). This approach had some limitations. It assumes investors are rational, markets are efficient, and asset returns are normally distributed. As a response to the some limitations of Markowitz theory minimum-VaR approach was appeared. This theory recognizes some assymetry, that investors are more concerned about potential losses than gains and incorporates downside risk measures like Value-at-Risk.

Despite advancements of the classical Markowitz theory and minimum VaR approach, challenges remain in accurately estimating parameters, singularity of the covariance matrix and managing risks in volatile markets.

In this paper we consider the mean-variance and mean-Var optimal portfolios and take into account the case when the covariance estimated matrix is singular. We use the Moore-Penrose pseudoinverse and Singular Value Decomposition (SVD) to find solutions. We apply these approaches and methodics to real financial data, construct mean-variance and mean-Var optimal portfolios and compare the dynamics of expected returns (mean), volatility and VaR for it.

Thanks to the proposed approaches, the investor gets a tool that allows him to make decisions about choosing an approach to building an optimal portfolio, as well as taking into account the singularity of the covariance matrix.

Keywords: portfolio optimization, mean-variance analysis, Markowitz optimal portfolio, value-at-risk (VaR), min-VaR analysis, Moore-Penrose pseudoinverse, parameter estimation.

Introduction

Portfolio optimization is a critical area in finance that focuses on selecting the best mix of assets to maximize returns while minimizing risk. The history of portfolio optimization is rich and has evolved significantly over the decades. The idea of diversification dates back centuries, with early investors recognizing that holding a variety of investments could reduce risk. The formalization of portfolio optimization began with Harry Markowitz's seminal paper "Portfolio Selection," published in 1952 [8]. Markowitz introduced the concept of mean-variance optimization, which quantifies the trade-off between risk (variance) and return (expected return). Moreover, Markowitz developed the concept of the efficient frontier, a graphical representation of optimal portfolios that offer the highest expected return for a given level of risk.

However Markowitz theory had some limitations. It assumes investors are rational and risk-averse, markets are efficient, and asset returns are normally distributed. Beside this, Markowitz mean-variance optimization requires inverting the covariance matrix to find the optimal portfolio weights. If the matrix is singular, there are either infinitely many solutions or none, and the standard approach fails.

As a response to the some limitations of Markowitz theory minimum-VaR theory was appeared [1],[2]. It incorporates downside risk measures like Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR)[7], [10]. This theory recognizes some assymetry, that investors are more concerned about potential losses than gains, leading to different optimization approaches [3]. In [5] estimators, confidence regions, and test for minimum VaR and CVaR optimal portfolios were considered.

Despite advancements of Markowitz and minimum-VaR theories, the challenges remain in accurately estimating parameters, managing risks in volatile markets, and adapting to changing economic conditions.

If the estimate covariance matrix is singular, there are either infinitely many solutions or none, and the standard approaches fail. In this case was proposed use the Moore-Penrose pseudoinverse or Singular Value Decomposition (SVD) to find solutions [4]. These methods allow for a solution that minimizes the L2-norm (the sum of squared portfolio weights), providing a unique and stable result among the infinite possibilities [3].

In this paper we focus on the application of mean-variance and min-Var optimal portfolios to real data and take into account the case when the covariance estimated matrix is singular. We use the Moore-Penrose pseudoinverse and Singular Value Decomposition (SVD) to find solutions. We apply this methodic to real financial data, construct mean-variance and min-VaR optimal portfolios and compare the dynamics of expected returns (means), volatility and VaR for it.

The mean-variance and min-VaR optimal portfolios: construction

The mean-variance portfolio. Let $x_t = (x_{1t}, \dots, x_{kt})'$ be a vector of returns for k-dimensional risky assets at time point $t = 1, \dots, n$. We assume that $x_1, \dots, x_n$ are independently and identically normally distributed, with a mean vector μ and covariance matrix Σ. We also assume that Σ may be singular, with $rank(\Sigma) = r_n < n < k + 1$.

Furthermore, let $w = (w_1, \dots, w_k)'$ be a k-dimensional vector of portfolio weights, where w_i represents the portion of wealth allocated to the i-th asset and $1_k' w = 1$, where 1_k stands for the k-dimensional vector of ones.

We denote the expected return and variance of the portfolio by $R = w'\mu$ and $V = w'\Sigma w$, respectively.

Following the classical mean-variance expected utility (EU) approach introduced by Markowitz, the optimal portfolio maximizes the trade-off between expected return and risk (measured as variance). The optimization problem is formulated as:

$$\max_{\mathbf{w}}[\mu_w - \frac{\tau}{2} \cdot \sigma_w^2] = \sum_{i=1}^{k} \mu_i w_i - \frac{\tau}{2} \left(\sum_{i=1}^{k} \sum_{j=1}^{k} \sigma_{ij} w_i w_j \right) \tag{1}$$

subject to

$$\sum_{i=1}^{k} w_i = 1$$

where $\tau > 0$ is the risk-aversion parameter, which reflects the investor's tolerance to risk. A larger value of τ implies that the investor places more weight on minimizing risk, while a smaller value emphasizes return maximization. The closed-form solution to the optimization problem (1) is given by:

$$\mathbf{w} = \frac{\Sigma^{-1}\mathbf{1}_k}{\mathbf{1}_k^T \Sigma^{-1} \mathbf{1}_k} + \frac{1}{\tau} R\mu \tag{2}$$

where R is the projection matrix that orthogonalizes the return vector μ with respect to the constraint $\mathbf{1}_k' x = 1$:

$$R = \Sigma^{-1} - \frac{\Sigma^{-1}\mathbf{1}_k \mathbf{1}_k^T \Sigma^{-1}}{\mathbf{1}_k^T \Sigma^{-1} \mathbf{1}_k} \tag{3}$$

and $\mathbf{1}_k$ is a k-dimensional vector of ones.

The optimal portfolios (EU) as proposed by Markowitz's theory lie on the upper part of the parabola in the mean-variance space. This parabola is known as the efficient frontier (EF) and, if Σ is positive definite, is given by

$$(R - R_{\text{GMV}})^2 = s(V - V_{\text{GMV}})$$

where

$$R_{\text{GMV}} = \frac{\mathbf{1}_k' \Sigma^{-1} \mu}{\mathbf{1}_k' \Sigma^{-1} \mathbf{1}_k} \tag{4}$$

and

$$V_{\text{GMV}} = \frac{1}{\mathbf{1}_k' \Sigma^{-1} \mathbf{1}_k} \tag{5}$$

are the expected return and variance of the global minimum variance portfolio ($GMVP$) given by (see, e.g., [6]) with parameter

$$s = \mu' R \mu,$$

where R is defined by 3

Thus, for constructing the optimal portfolio following the classical mean-variance expected utility (EU) approach introduced by Markowitz, we need just to compute the weights by 2. The expected return (mean) R of the EU optimal portfolio one can compute by 3, the variance is defined by $V = w'\Sigma w$ and Σ is positive definite.

The minimum-VaR portfolio. Markowitz theory assumes investors are rational and risk-averse, markets are efficient, and asset returns are normally distributed.

Nevertheless real world admits some assymetry, investors are more concerned about potential losses than gains. In the papers [1; 2] were proposed to use Value-at-Risk (VaR) as risk measures in Markowitz's optimization problem, instead of the traditional variance.

VaR is defined as the potential loss of an investment portfolio at a given confidence level. This measure is believed to provide a more accurate representation of the risk in investor problem [11] and portfolio management.

Formally, the value-at-risk of level α, $0 < \alpha \leq 1$ is a probability functional, defined as α-quantile of the profit (loss) function Y

$$V@R_\alpha(Y) = G^{-1}(\alpha) = inf\{\ y \in R : G(Y) \geq \alpha\}\ ,$$

where G is the distribution function of Y, G^{-1} is the quantile function of α, $0 < \alpha \leq 1$.

It is worth to note that often it is recommended (for examples by regulators Basel I and Basel II) to denote VaR as the low quantile with minus sign:

$$V@R_\alpha(Y) = -G^{-1}(\alpha)$$

For portfolio analysis we use the rate of return x_w as the profit (loss) function Y. Then the VaR at the confidence level $\alpha \in (0.5, 1)$ (VaR_α) is defined as the rate of return x_w such that

$$P\left\{x_w < -\mathrm{VaR}_\alpha\right\} = 1 - \alpha \qquad (6)$$

where

$$x_w = x'w.$$

The optimization problem, as proposed by [1; 2], can be stated as follows:

$$VaR_\alpha \to \min, \quad \text{subject to} \quad \mathbf{1}_k' w = 1. \qquad (7)$$

[1; 2] have derived the exact expressions of its weights and characteristics. In addition, they have shown that the necessary and sufficient condition for constructing the minimum VaR portfolio, i.e., for a solution to exist in (7), is $s < z_\alpha^2$. Here, the quantity $z_\alpha = -\Phi^{-1}(1-\alpha)$ denotes the α-quantile of the standard normal distribution.

In the paper [5] were introduced and used alternative expressions of the weights and of the characteristics of the minimum VaR portfolio in terms of (4) and (5). The weights of the portfolio obtained in (7) are given by w_{VaR}:

$$w_{VaR} = w_{\mathrm{GMV}} + \frac{\sqrt{V_{\mathrm{GMV}}}}{\sqrt{z_\alpha^2 - s}} R\mu, \qquad (8)$$

where

$$w_{GMV} = \frac{\Sigma^{-1}1_k}{\mathbf{1}_k'\Sigma^{-1}\mathbf{1}_k}.$$

The portfolio's value-at-risk is M_{VaR}, with a mean of R_{VaR} and variance of V_{VaR}:

$$M_{\mathrm{VaR}} = \sqrt{z_\alpha^2 - s}\sqrt{V_{\mathrm{GMV}}} - R_{\mathrm{GMV}}.$$

$$R_{\mathrm{VaR}} = w_{\mathrm{VaR}}'\mu = R_{\mathrm{GMV}} + \frac{s}{\sqrt{z_\alpha^2 - s}}\sqrt{V_{\mathrm{GMV}}},$$

Thus, for constructing the min-VaR optimal portfolio we need to compute the weights w_{VaR} by 8. The expected return (mean) R_{VaR} in this case one can compute by 8, the variance is defined by $V = w'\Sigma w$ and Σ is positive definite.

The VaR evaluation. In this section we discuss in more details the problems of VaR estimation.

For evaluating VaR there are some methods. VaR can be estimated either parametrically (for example, variance-covariance VaR) or non-parametrically (for examples, historical simulation VaR or resampled VaR). A McKinsey report published in May 2012 estimated that 85% of large banks were using historical simulation and the other 15% used Monte Carlo methods. We can notice, that in [13] we applied the Markowitz technics to construct the optimal portfolio for real data. Moreover, we apply Monte Carlo method to compute VaR for constructed portfolios with some assumption of their distribution. In [12], [14] we evaluate VaR by parametric method as α-quantile of the loss-profit function G with known parameters. In this paper we focus on historical and non-parametric methods.

Historical (non-parametric) method. This is the most intuitive approach, relying solely on historical return data without any distributional assumptions. The method involves the following steps:

- Sort the historical portfolio returns x_w in ascending order.
- Identify the quantile corresponding to the loss level $\alpha \in (0, 1)$, which corresponds to the confidence level $(1 - \alpha)$.

The VaR is defined by 6 and estimated as empirical $(1 - \alpha)$-quantile of the sorted sample x_w.

Parametric (variance-covariance) method. This method assumes that the returns of the asset or portfolio are normally distributed. Given the standard deviation σ and the portfolio value W, the VaR for a single period is calculated as:

$$VaR_\alpha = z_\alpha \cdot \sigma \cdot W,$$

where $z_\alpha = -\Phi^{-1}(1 - \alpha)$ is the standard normal quantile corresponding to the specified loss level α.

For a multi-period horizon of length t, the formula becomes:

$$VaR_\alpha(t) = z_\alpha \cdot \sigma \cdot W \cdot \sqrt{t},$$

assuming the returns are independent and identically distributed across time.

Commonly used values of α include 0.1, 0.05, and 0.01, which correspond to confidence levels of 90%, 95%, and 99%, respectively. The associated quantiles z_α are summarized below:

Confidence level	z_α
90%	1.282
95%	1.645
99%	2.326

Example 1. Let us estimate the VaR of a previously constructed portfolio using the parametric method. Consider the following inputs:

$$\Sigma = \begin{bmatrix} 0.05 & 0.01 & 0.02 \\ 0.01 & 0.04 & 0.015 \\ 0.02 & 0.015 & 0.03 \end{bmatrix},$$

$$\mu = \begin{bmatrix} 0.15 \\ 0.10 \\ 0.12 \end{bmatrix},$$

$$\alpha = 0.95, \quad W = 1$$

Portfolio weights: $[0.289,\ 0.289,\ 0.422]$
First, we compute the portfolio variance:

$$\sigma = \sqrt{0.02307} \approx 0.1519$$

Then, the 95% Value-at-Risk is calculated as:

$$VaR_{0.95} = 1.645 \times 0.1519 \approx 0.250$$

Hence, with 95% confidence, the maximum expected portfolio loss over the period is approximately not more than 25% of the portfolio value.

Estimators: non-singular and singular cases

In practice Σ is an unknown matrix and should be estimated using historical values of asset returns. Given a sample of n independent observations $x_1, \ldots, x_n$ of returns on k assets we calculate the sample estimators of μ – the mean vector and Σ – the covariance matrix, respectively by

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad S = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(x_i - \overline{x})'.$$

Non-singular case. If the sample covariance matrix S is non-singular, then the formulas for constructing mean-variance portfolio or minimal VaR portfolio can be applied directly by replacing the unknown population covariance matrix Σ with the sample covariance matrix S.

Example 2: Solving the Portfolio Optimization Problem

Given the covariance matrix Σ, expected returns vector μ, and risk aversion parameter τ:

$$\Sigma = \begin{bmatrix} 0.05 & 0.01 & 0.02 \\ 0.01 & 0.04 & 0.015 \\ 0.02 & 0.015 & 0.03 \end{bmatrix}, \quad \mu = \begin{bmatrix} 0.15 \\ 0.1 \\ 0.12 \end{bmatrix}, \quad \tau = 10$$

The inverse of the covariance matrix is:

$$\Sigma^{-1} = \begin{bmatrix} 27.27 & 0 & -18.18 \\ 0 & 30.77 & -15.38 \\ -18.18 & -15.38 & 53.15 \end{bmatrix}$$

The first term of the solution:

$$\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} = \begin{bmatrix} 0.206 \\ 0.349 \\ 0.445 \end{bmatrix}$$

The second term:

$$\frac{1}{\tau}R\mu = \begin{bmatrix} 0.0829 \\ -0.0600 \\ -0.0222 \end{bmatrix}$$

Final portfolio weights:

$$\mathbf{x} = \begin{bmatrix} 0.289 \\ 0.289 \\ 0.422 \end{bmatrix}$$

Python Implementation. The function `optimize_portfolio_by_Markowitz_2_test` takes the following parameters:

- `mean_returns` – vector of expected returns
- `cov_matrix` – covariance matrix
- `T` – risk aversion parameter τ

```python
def optimize_portfolio_by_Markowitz_2_test(mean_returns=None, cov_matrix=None, T=0):
    cov_matrix_inv = np.linalg.solve(cov_matrix, np.eye(cov_matrix.shape[0]))
    ones = np.ones_like(mean_returns)
    divider = ones.T @ cov_matrix_inv @ ones

    # First term
    first_term = (cov_matrix_inv @ ones) / divider

    # Second term
    R = cov_matrix_inv - (np.outer(cov_matrix_inv @ ones, ones) @ cov_matrix_inv) / divider
    second_term = (R @ mean_returns) / T

    result = first_term + second_term
    return {'x': result, 'fun': calculate_volatility(result, cov_matrix)}
```

Figure 1. Function implementation of Markowitz Model 2

The function uses `numpy.linalg.solve` to compute the inverse, constructs a vector of ones, and calculates the optimal weights according to the extended Markowitz model.

```
Covariance matrix:
[[0.05  0.01  0.02 ]
 [0.01  0.04  0.015]
 [0.02  0.015 0.03 ]]

Markowitz (T = 10):
Optimal asset weights: [0.289, 0.289, 0.422]
Volatility: 0.1518771433319269
VaR (Parametric method): 0.249816
```

Figure 2. Resulting optimal portfolio weights

This example demonstrates how to build an optimal portfolio that balances expected return and risk using the extended Markowitz model.

Singular case. In practical applications, the sample covariance matrix S may be singular. When S is nonsingular, it is possible to use the Moore-Penrose pseudoinverse instead of the regular matrix inverse [4; 9].

The *Moore-Penrose pseudoinverse* of a matrix $A \in \mathbb{R}^{m\times n}$, denoted by A^+, is defined as the unique matrix that satisfies the following four conditions:

$$\begin{aligned} AA^+A &= A, \\ A^+AA^+ &= A^+, \\ (AA^+)^\top &= AA^+, \\ (A^+A)^\top &= A^+A. \end{aligned}$$

This matrix generalizes the concept of an inverse to possibly singular or non-square matrices.

To compute the Moore-Penrose pseudoinverse in practice, one typically uses the Singular Value Decomposition (SVD). Given a matrix A of size $m \times n$, it can be decomposed as:

$$A = U\Sigma V^\top,$$

where U and V are orthogonal matrices, and Σ is a diagonal matrix with non-negative singular values. The pseudoinverse is then given by:

$$A^+ = V\Sigma^+U^\top,$$

where Σ^+ is obtained by taking the reciprocal of the non-zero entries of Σ and transposing the resulting matrix.

Example 3.

In this example we would like to demonstrate how to compute the inverted matrix by hands and by Python. Consider the singular matrix:

$$A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$$

Compute the determinant:

$$\begin{aligned} \det(A) &= 3 \cdot (-2) - 6 \cdot (-1) \\ &= -6 + 6 = 0 \end{aligned}$$

Since $\det(A) = 0$, matrix A is singular and cannot be inverted classically.

SVD decomposition:

$$A = U\Sigma V^T$$

Singular values:

$$\sigma_1 = \sqrt{50}, \quad \sigma_2 = 0$$

$$\Sigma = \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma^+ = \begin{bmatrix} \frac{1}{\sqrt{50}} & 0 \\ 0 & 0 \end{bmatrix}$$

Matrices U and V:

$$U = \begin{bmatrix} -0.9487 & -0.3162 \\ 0.3162 & -0.9487 \end{bmatrix}, \quad V = \begin{bmatrix} -0.4472 & -0.8944 \\ -0.8944 & 0.4472 \end{bmatrix}$$

Transpose of U:

$$U^T = \begin{bmatrix} -0.9487 & 0.3162 \\ -0.3162 & -0.9487 \end{bmatrix}$$

Intermediate multiplication:

$$\Sigma^+U^T = \begin{bmatrix} \frac{1}{\sqrt{50}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.9487 & 0.3162 \\ -0.3162 & -0.9487 \end{bmatrix} = \begin{bmatrix} -0.1341 & 0.0447 \\ 0 & 0 \end{bmatrix}$$

Final multiplication:

$$A^+ = \begin{bmatrix} -0.4472 & -0.8944 \\ -0.8944 & 0.4472 \end{bmatrix} \begin{bmatrix} -0.1341 & 0.0447 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.06 & -0.02 \\ 0.12 & -0.04 \end{bmatrix}$$

Final result:

$$A^+ = \begin{bmatrix} 0.06 & -0.02 \\ 0.12 & -0.04 \end{bmatrix}$$

Now, consider the same example using a Python implementation and verify the result.

We use the same matrix:

$$A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$$

To verify the manual result, we compute the pseudoinverse using Python.

Determinant:

$$\det(A) = 0.0$$

Using `np.linalg.pinv`:

$$A^+ = \begin{bmatrix} 0.06 & -0.02 \\ 0.12 & -0.04 \end{bmatrix}$$

Using `np.linalg.svd` and manual reconstruction:

$$A^+ = \begin{bmatrix} 0.06 & -0.02 \\ 0.12 & -0.04 \end{bmatrix}.$$

To construct *mean-variance optimal portfolio* in the case of a singular sample covariance matrix S, the portfolio weights under the expected utility criterion are estimated using:

$$\hat{\mathbf{w}}^+_{EU} = \frac{\mathbf{S}^+\mathbf{1}_k}{\mathbf{1}_k^\top \mathbf{S}^+\mathbf{1}_k} + \alpha^{-1}\widehat{\mathbf{R}}^+\bar{\mathbf{x}},$$

where

$$\widehat{\mathbf{R}}^+ = \mathbf{S}^+ - \frac{\mathbf{S}^+\mathbf{1}_k\mathbf{1}_k^\top\mathbf{S}^+}{\mathbf{1}_k^\top\mathbf{S}^+\mathbf{1}_k}.$$

Similarly, *the Global Minimum Variance (GMV)* portfolio estimators in the singular case are obtained as:

$$\begin{aligned} \hat{R}^+_{\text{GMV}} &= \frac{\mathbf{1}_k^\top S^+\,\overline{x}}{\mathbf{1}_k^\top S^+\mathbf{1}_k}, \\ \hat{V}^+_{\text{GMV}} &= \frac{1}{\mathbf{1}_k^\top S^+\mathbf{1}_k}, \\ \hat{\mathbf{w}}^+_{\text{GMV}} &= \frac{S^+\mathbf{1}_k}{\mathbf{1}_k^\top S^+\mathbf{1}_k}. \end{aligned}$$

To construct *the Value-at-Risk (VaR) efficient portfolio* in the singular case we compute the portfolio weights in the form:

$$\hat{w}^+_{\text{VaR}} = \hat{w}^+_{\text{GMV}} + \frac{\sqrt{\hat{V}^+_{\text{GMV}}}}{\sqrt{z^2_\alpha - \hat{s}^+}} \cdot \hat{R}^+ \bar{x}.$$

where

$$\hat{s}^+ = \bar{x}^\top \hat{R}^+ \bar{x}, \quad \text{and} \quad \hat{R}^+ = S^+ - \frac{S^+ 1_k 1_k^\top S^+}{1_k^\top S^+ 1_k}.$$

Then, for singular case the portfolio's value-at-risk, the estimated VaR-efficient return and variance are given by:

$$\hat{M}^+_{\text{VaR}} = \sqrt{z^2_\alpha - \hat{s}^+} \cdot \sqrt{\hat{V}^+_{\text{GMV}}} - \hat{R}^+_{\text{GMV}},$$

$$\hat{R}^+_{\text{VaR}} = \hat{R}^+_{\text{GMV}} + \frac{\hat{s}^+}{\sqrt{z^2_\alpha - \hat{s}^+}} \cdot \sqrt{\hat{V}^+_{\text{GMV}}},$$

$$\hat{V}^+_{\text{VaR}} = \frac{z^2_\alpha}{z^2_\alpha - \hat{s}^+} \cdot \hat{V}^+_{\text{GMV}}.$$

Example 4. In this example we would like to demonstrate how to construct Marcowitz portfolio in the singular case by hands and by Python. Now we consider an example of Markowitz portfolio optimization with an investor risk aversion coefficient $\tau = 4000$. This case illustrates a singularity scenario since the covariance matrix has a zero determinant. The Singular Value Decomposition (SVD) algorithm is demonstrated along with step-by-step calculations.

$$\mu_1 = 0.1, \quad \mu_2 = 0.08, \quad \tau = 4000$$

$$\Sigma = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

To compute the Moore–Penrose pseudoinverse of Σ, we proceed as follows:

$$\Sigma\Sigma^T = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix}$$

$$\det \begin{bmatrix} 20-\lambda & 10 \\ 10 & 5-\lambda \end{bmatrix} = (20-\lambda)(5-\lambda)-100 = \lambda^2 - 25\lambda = 0$$

$$\lambda_1 = 25, \quad \lambda_2 = 0, \quad \sigma_1 = 5, \quad \sigma_2 = 0$$

$$\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma^+ = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{bmatrix}$$

For the eigenvectors of $\Sigma\Sigma^T$, we have:

$$(\Sigma\Sigma^T - 25I)\vec{u} = 0 \Rightarrow \begin{bmatrix} -5 & 10 \\ 10 & -20 \end{bmatrix} \vec{u}_1 = 0 \Rightarrow$$

$$\hat{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(\Sigma\Sigma^T - 0I)\vec{u} = 0 \Rightarrow \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \vec{u}_2 = 0 \Rightarrow$$

$$\hat{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$v_1 = \frac{1}{5}\Sigma^T u_1 = \frac{1}{5\sqrt{5}} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$V\Sigma^+ = \begin{bmatrix} \frac{1}{5\sqrt{5}} & 0 \\ \frac{2}{5\sqrt{5}} & 0 \end{bmatrix}$$

$$U^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$\Sigma^+ = V\Sigma^+ U^T = \begin{bmatrix} \frac{2}{25} & \frac{1}{25} \\ \frac{4}{25} & \frac{2}{25} \end{bmatrix}$$

Next, we calculate the optimal portfolio using this pseudoinverse:

$$\Sigma^+ = \begin{bmatrix} 0.08 & 0.04 \\ 0.16 & 0.08 \end{bmatrix}, \quad \mu = \begin{bmatrix} 0.1 \\ 0.08 \end{bmatrix}, \quad \mathbf{1}_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Sigma^+ \mathbf{1}_k = \begin{bmatrix} 0.12 \\ 0.24 \end{bmatrix}, \quad \mathbf{1}_k^\top \Sigma^+ \mathbf{1}_k = 0.36$$

$$\frac{\Sigma^+ \mathbf{1}_k}{\mathbf{1}_k^\top \Sigma^+ \mathbf{1}_k} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$\Sigma^+ \mu = \begin{bmatrix} 0.012 \\ 0.024 \end{bmatrix}, \quad \frac{1}{2\tau} = \frac{1}{8000}$$

$$\frac{1}{8000} \cdot \Sigma^+ \mu = \begin{bmatrix} 1.5 \times 10^{-6} \\ 3 \times 10^{-6} \end{bmatrix}$$

$$w_{EU} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} + \begin{bmatrix} 1.5 \times 10^{-6} \\ 3 \times 10^{-6} \end{bmatrix} = \begin{bmatrix} 0.3330015 \\ 0.666003 \end{bmatrix}$$

Now we replicate the same example using a Python program to verify the correctness of the manual computations. The result is a vector x

representing the optimal asset weights that simultaneously account for risk minimization, expected return, and the investor's risk aversion. For clarity, the program also prints auxiliary results such as the pseudoinverse matrix Σ^+, individual components $x^{(1)}$, $x^{(2)}$, and the final portfolio weights.

We use the same input:

$$\mu_1 = 0.1, \quad \mu_2 = 0.08, \quad \tau = 4000, \quad \Sigma = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

The output of the Python implementation is:

$$\det(\Sigma) = 0.0$$

$$\Sigma^+ = \begin{bmatrix} 0.08 & 0.04 \\ 0.16 & 0.08 \end{bmatrix}$$

$$\text{Minimum variance component} = \begin{bmatrix} 0.33333333 \\ 0.66666667 \end{bmatrix},$$

$$\text{Risk-adjusted return component} = \begin{bmatrix} 1 \cdot 10^{-7} \\ 2 \cdot 10^{-7} \end{bmatrix}$$

$$\text{Optimal portfolio weights} = \begin{bmatrix} 0.33333343 \\ 0.66666687 \end{bmatrix}$$

These results confirm the accuracy of both the manual and programmatic approaches to portfolio optimization in the presence of a singular covariance matrix.

Illustration for the Real Financial Data

Markowitz Portfolio. The data for analysis was obtained from the Yahoo Finance API using the Python library `yfinance`. For each asset, historical daily closing prices were downloaded for the period from November 30, 2022 to November 30, 2023.

The portfolio consists of five assets:

- Amazon (AMZN) - Technology
- Mastercard (MA) - Finance
- Netflix (NFLX) - Entertainment
- Uber (UBER) - Transport
- Adobe (ADBE) - Software

Two values of the τ parameter were selected for the study:

- $\tau = 20$ — aggressive strategy focusing on maximizing return
- $\tau = 100$ — conservative strategy aiming at minimizing risk

Parameter	**Value**
Optimal weights	
AMZN	23.2%
MA	2.3%
NFLX	43.9%
UBER	10.0%
ADBE	20.6%
Portfolio volatility	0.01292
Return distribution	Not normal
VaR (95%)	
Parametric	0.336805
Historical	0.020637

Table 1. Portfolio characteristics for $\tau = 20$

Parameter	**Value**
Optimal weights	
AMZN	5.5%
MA	4.0%
NFLX	78.7%
UBER	5.7%
ADBE	6.1%
Portfolio volatility	0.01118
Return distribution	Not normal
VaR (95%)	
Parametric	0.291376
Historical	0.017616

Table 2. Portfolio characteristics for $\tau = 100$

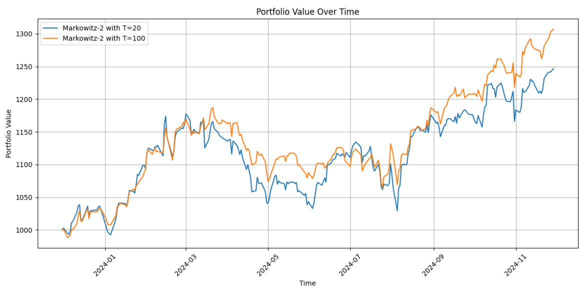

Figure 3. Portfolio wealth dynamics for Markowitz portfolio with $\tau = 20$ and $\tau = 100$

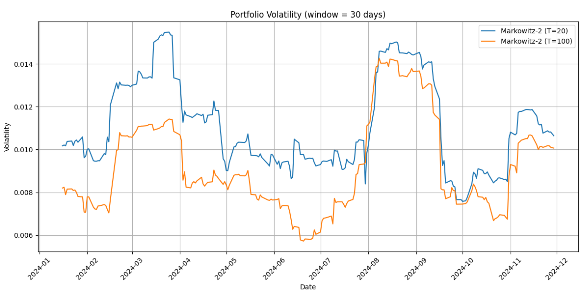

Figure 4. Parametric VaR dynamics for Markowitz portfolio with $\tau = 20$ and $\tau = 100$

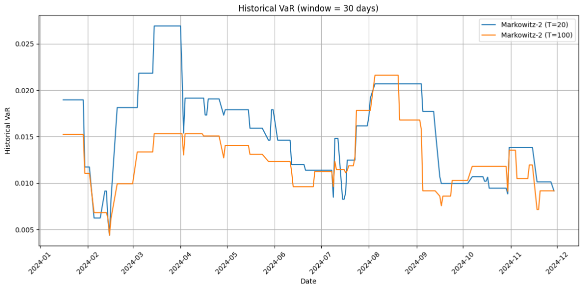

Figure 5. Historical VaR dynamics for Markowitz portfolio with $\tau = 20$ and $\tau = 100$

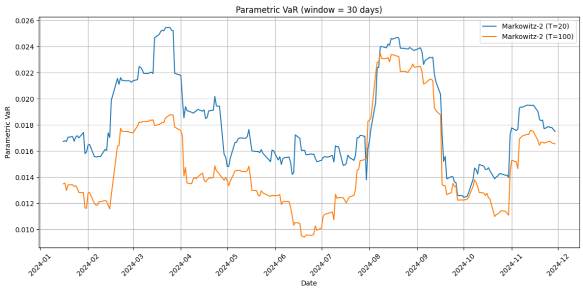

Figure 6. Portfolio wealth dynamics for Markowitz portfolio with $\tau = 20$ and $\tau = 100$

Our analysis highlights how varying the τ parameter impacts portfolio behavior. With $\tau = 20$, the portfolio remains diversified—43.9% in NFLX, 23.2% in AMZN, and 20.6% in ADBE. A shift to a conservative $\tau = 100$ results in a highly concentrated allocation (78.7% in NFLX) and reduces volatility by 13.5%.

This is confirmed by declines in both parametric VaR (from 0.337 to 0.291) and historical VaR (from 0.021 to 0.018), indicating lower risk. Given the non-normal return distribution, historical VaR proves more reliable.

Overall, the analysis shows how adjusting τ balances concentration and risk, linking theoretical models to real market dynamics. The findings underscore the importance of strategy calibration for achieving targeted risk-return profiles.

Minimum VaR portfolio. The data for this analysis was obtained using the same methodology as in the previous example, sourced from the Yahoo Finance API via the `yfinance` Python library. The portfolio comprises the same five assets: Amazon (AMZN), Mastercard (MA), Netflix (NFLX), Uber (UBER), and Adobe (ADBE), with historical daily closing prices covering the period from November 30, 2022, to November 30, 2023.

The optimal weights for the Minimum VaR Portfolio were calculated to minimize the Value-at-Risk (VaR). The resulting allocation is:

Parameter	**Value**
Optimal weights	
AMZN	25.59%
MA	0.14%
NFLX	42.56%
UBER	31.71%
ADBE	0.00%
Portfolio volatility	0.01379
Return distribution	Not normal
VaR (95%)	
Parametric	0.359239
Historical	0.019444

Table 3. Key characteristics of the VaR-min portfolio.

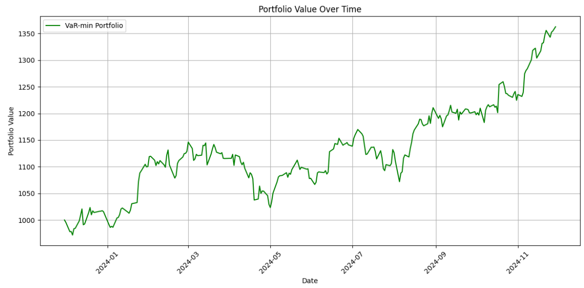

Figure 7. Portfolio wealth dynamics for the min-VaR portfolio.

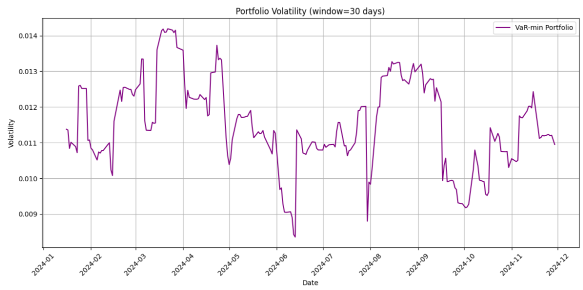

Figure 8. Portfolio volatility dynamics for the min-VaR portfolio.

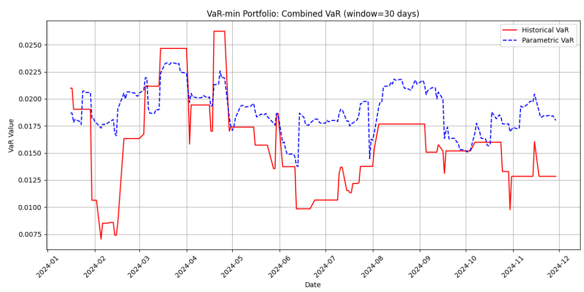

Figure 9. Historical and parametric VaR dynamics for the min-VaR portfolio.

The analysis demonstrates that the minimum

VaR portfolio is predominantly allocated to Netflix (NFLX) and Uber (UBER), with only marginal exposure to Mastercard (MA) and Adobe (ADBE). This allocation pattern reflects the optimization objective of minimizing Value-at-Risk (VaR), which naturally favors assets that have shown greater stability during adverse market conditions.

Compared to the conservative Markowitz portfolio ($\tau = 100$), the VaR-min portfolio exhibits slightly higher volatility. However, it achieves marginally lower historical VaR, highlighting its superior resilience during periods of market stress. Given the non-normal nature of return distributions, historical VaR emerges as a more reliable indicator of downside risk than the parametric measure.

Conclusion and discussion

In the paper two approaches to portfolio optimization were considered: the mean-variance and min-VaR technics. We apply these approaches to real financial data.

Firstly we estimate the parameters and take into account the case when the covariance estimated matrix is singular. We use the Moore-Penrose pseudoinverse and Singular Value Decomposition (SVD) to find solutions and demonstrate this in terms of some example.

After that we construct mean-variance and min-VaR optimal portfolios and compare the dynamics of portfolio wealth (means), volatility and Value-ar-risk for it.

The analysis provided in Python demonstrates some interesting facts. For Marcowitz portfolio we see how adjusting τ balances concentration and risk, linking theoretical models to real market dynamics. The findings underscore the importance of strategy calibration for achieving targeted risk-return profiles.

Minimum VaR allocation pattern reflects the optimization objective of minimizing Value-at-Risk (VaR), which naturally favors assets that have shown greater stability during adverse market conditions.

The time dynamics of portfolio mean and VaR confirm the effectiveness of min-VaR strategy in balancing risk and return. By prioritizing VaR minimization over traditional mean-variance objectives, the resulting portfolio provides a distinct and practical alternative framework for portfolio construction.

This min-VaR approach underscores the value of targeted optimization in modern portfolio theory: by explicitly focusing on risk protection, investors can achieve portfolios that are not only theoretically sound but also better aligned with real-world risk management objectives.

Theoretical researches (see for example [1]) show that the min-VaR optimization problem is equivalent to Markowitz's optimization problem if the returns on assets are multivariate normally distributed. As a result, all optimal portfolios obtained by solving (7) are lying on the EF, the set of optimal portfolios resulting from Markowitz's approach. The returns on real assets are not mostly multivariate normally distributed. It is a reason why the min-VaR and mean-variance portfolios demonstrate differebt behavour for real data.

Thanks to the analysis to proposed approaches, the investor gets a tool that allows him to make decisions about choosing an approach to building an optimal portfolio, as well as taking into account the singularity of the covariance matrix.

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ПОРТФЕЛЬНА ОПТИМІЗАЦІЯ ДЛЯ РЕАЛЬНИХ ДАНИХ: ПІДХОДИ ТА ВИКЛИКИ

Теорія портфельної оптимізації продовжує бути динамічною галуззю у фінансах, інтегруючи нові теорії та підходи для кращого задоволення потреб інвесторів. З розвитком фінансових ринків розвиватимуться й нові підходи для оптимізації портфелів, що робить цей напрям сприятливим для появи нових досліджень та інновацій.

Класичний підхід Марковіца базується на оптимізації функції, яка кількісно визначає компроміс між ризиком (дисперсією) та очікуваною дохідністю. Проте цей підхід має деякі обмеження. Зокрема, він припускає, що інвестори раціональні, їхнє ставлення до ризику регулюється деяким параметром, ринки ефективні, а дохідність активів розподілена нормально. У відповідь на обмеження теорії Марковіца з'явився інший підхід, що визнає певну асиметрію, тобто інвестори більше стурбовані потенційними збитками, ніж прибутками. Цей підхід базується на мінімізації так званого показника величини ризику Value-at-Risk. Незважаючи на досягнення класичної теорії Марковіца та підходу мінімізації VaR-показника, залишаються виклики, пов'язані з проблемами оцінки параметрів, можливістю появи сингулярної оціночної коваріаційної матриці та управлінням ризиками на волатильних ринках.

У цій статті ми розглядаємо побудову оптимальних портфелів як за підходом Марковіца, так і за мінімізацією показника величини ризику, а також враховуємо випадок, коли коваріаційна оціночна матриця є сингулярною. Ми використовуємо псевдообернений метод Мура—Пенроуза та розкладання за сингулярним значенням (SVD) для пошуку рішень. Ми застосовуємо ці підходи та методики до реальних фінансових даних, будуємо оптимальні портфелі за двома підходами, порівнюємо динаміку зміни дохідності, варіатичності і показника ризику для цих оптимальних портфелей між собою і з динамікою рівномірного портфеля.

Завдяки запропонованим підходам інвестор отримує інструмент, який дозволяє йому приймати рішення щодо вибору підходу при побудові оптимального портфеля, а також враховувати сингулярність коваріаційної матриці.

Ключові слова: оптимізація портфеля інвестора, середньодисперсійний аналіз, оптимальний портфель Марковіца, вартісна міра ризику (VaR), мінімум-VaR-аналіз, псевдообернений метод Мура—Пенроуза, оцінка параметрів.

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