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LAST TIME MOMENT OPTIMALITY IN UNIFORM 1-BULLET SILENT DUEL WITH SCALED EXPONENTIALLY-CONVEX ACCURACY

The uniform 1-bullet silent duel with scaled exponentially-convex accuracy of payoffs is a symmetric matrix game whose optimal value is 0, and each of the duelists has the same optimal behavior, whether it is in pure or mixed strategies. Such duels model two-side competitive interaction, where the purpose is to gain a reward by making the best possible decision through quantized time. It is proved that the last time moment is optimal in the duel with N time moments only when the accuracy factor does not exceed marginal value $\frac{e-e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}}-1}$. If the accuracy factor is dropped below this marginal value, then the last time moment is single optimal. If the accuracy factor is exactly equal to the marginal value, the duelist has two optimal time moments: the penultimate and last one. The conditions of the last time moment optimality can be set to force the duelist to act the latest possible, which is quite useful in some blockchain settings, where participants (e. g., validators or miners) choose when to attempt block proposal or transaction insertion under uncertainty.

Keywords: uniform 1-bullet silent duel, scaled accuracy, exponentially-convex accuracy, matrix game, last time moment optimality.

Uniform 1-bullet silent duels

The uniform 1-bullet silent duel is a timing game [1], [2]

$$\langle X_N,\ Y_N,\ \mathbf{U}_N\rangle = \\ = \left\langle \{x_i\}_{i=1}^{N},\ \{y_j\}_{j=1}^{N},\ \mathbf{U}_N \right\rangle \quad (1)$$

of two players (duelists) whose pure strategy sets are denoted by

$$X_N = \{x_i\}_{i=1}^{N} = \left\{\frac{i-1}{N-1}\right\}_{i=1}^{N} = T_N = \\ = \{t_q\}_{q=1}^{N} = \left\{\frac{q-1}{N-1}\right\}_{q=1}^{N} \subset [0;\ 1], \quad (2)$$

$$Y_N = \{y_j\}_{j=1}^{N} = \left\{\frac{j-1}{N-1}\right\}_{j=1}^{N} = T_N = \\ = \{t_q\}_{q=1}^{N} = \left\{\frac{q-1}{N-1}\right\}_{q=1}^{N} \subset [0;\ 1], \quad (3)$$

and payoff matrix $\mathbf{U}_N$ is skew-symmetric [3], [4]:

$$\mathbf{U}_N = [u_{ij}]_{N\times N} = [-u_{ji}]_{N\times N} = -\mathbf{U}_N^{\mathrm{T}}. \quad (4)$$

Uniform 1-bullet silent duels model two-side competitive interaction, where the purpose is to gain a reward by making the best possible decision through quantized time [2], [4]. This time is set T_N in (2), (3), consisting of N successive time moments $\{t_q\}_{q=1}^{N}$ of possible shooting [1] and representing the standardized time span [0; 1] upon its equidistant (uniform) quantization with a step of $\frac{1}{N-1}$. Thus, the duelist is allowed to legitimately shoot only at one of the time moments in set T_N, where number N determines the duel size. Shooting the bullet is a metaphor of making a single decision, where the duelist benefits from shooting as late as possible but only by shooting first [2], [3], [5].

Optimal time moment existence

Uniform 1-bullet silent duel (1) by (2) — (4), being a finite zero-sum game, always has a solution at least in mixed strategies [2], [3]. Besides, the duel is symmetric, and thus its optimal game value is 0, and each of the duelists has the same set of optimal strategies, which can be both pure and mixed [1], [6]. However, due to finite 1-bullet silent duels are commonly used to model non-repeatable interaction processes, the main goal is to determine all optimal time moments (optimal pure strategies) for the duelist to shoot [1], [7]. If there are no optimal time moments at the duelist, i. e. duel (1) is not solved in pure strategies (but, certainly, it is solved in mixed strategies), the duel configuration is forcedly modified through changing the structure of payoff matrix $\mathbf{U}_N$ in order to come up with a pure-strategy optimal behavior for the duelist [1].

Owing to the skew-symmetry of matrix (4), whose main diagonal is of N zeros, any saddle point of matrix (4) is a zero entry in a nonnegative row and a nonpositive column [3]. Thus, if row i^* of matrix (4) by $i^* \in \left\{\overline{1,\ N}\right\}$ is nonnegative, then row i^* contains a saddle point on the main diagonal [4] and time moment t_{i^*} is optimal. A symmetric reasoning is true for columns: if column i^* of matrix (4) by $i^* \in \left\{\overline{1,\ N}\right\}$ is nonpositive, then column i^* contains a saddle point on the main diagonal [2] and time moment t_{i^*} is optimal. So, it is conventionally possible to conclude on optimal time moment existence by studying only either nonnegative rows or nonpositive columns of payoff matrix (4).

If row i^* contains a negative entry, time moment t_{i^*} is not optimal. At that, column i^* contains the positive entry. If row i^* contains only positive entries, except for the main diagonal entry $u_{i^*i^*} = 0$, then there is the single optimal time moment t_{i^*} in this duel [4], [8].

Scaled payoff exponential rate

Duel (1) by (2) — (4) is configured by the structure of payoff matrix (4), which is determined by how its entries are calculated. In general,

$$u_{ij} = ag(x_i) - ag(y_j) + \\ +a^2 g(x_i)\, g(y_j)\, \mathrm{sign}\,(y_j - x_i) \\ \text{for } i = \overline{1,\ N} \text{ and } j = \overline{1,\ N} \quad (5)$$

by some discrete accuracy functions $g(x_i)$ and $g(y_j)$ of the first and second duelists, respectively, scaled with an accuracy factor $a > 0$, where

$$g(x_1) = g(y_1) = g(0) = 0 \text{ and} \\ g(x_N) = g(y_N) = g(1) = 1. \quad (6)$$

Commonly, these functions are nondecreasing. As rewards increase with time, and the increment is rather nonlinear than linear, it is appropriate to consider exponentially-increasing accuracy functions. So, instead of (5), entry u_{ij} of payoff matrix (4) is calculated as

$$u_{ij} = ag(e^{x_i}) - ag(e^{y_j}) + \\ +a^2 g(e^{x_i})\, g(e^{y_j})\, \mathrm{sign}\,(y_j - x_i) \\ \text{for } i = \overline{1,\ N} \text{ and } j = \overline{1,\ N} \quad (7)$$

by still obeying requirements similar to (6):

$$g(e^{x_1}) = g(e^{y_1}) = g\left(e^0\right) = g(1) = 0 \text{ and} \\ g(e^{x_N}) = g(e^{y_N}) = g\left(e^1\right) = g(e) = 1. \quad (8)$$

Seemingly, accuracy factor a just scales the reward, but its genuine impact will be ascertained below.

Assume that an exponentially-increasing accuracy function of the duelist is

$$g(e^z) = \alpha e^z + \beta \text{ by } \alpha \in \mathbb{R} \backslash \{0\},\ \beta \in \mathbb{R}. \quad (9)$$

As function (9) of variable z must obey requirements (8), then

$$g\left(e^0\right) = g(1) = \alpha + \beta = 0, \\ g\left(e^1\right) = g(e) = \alpha e + \beta = 1,$$

whence

$$\beta = -\alpha = 1 - \alpha e, \\ \alpha(e-1) = 1,$$

and

$$\alpha = \frac{1}{e-1},\ \ \beta = \frac{1}{1-e}. \quad (10)$$

Upon plugging (10) into (9) function $g(e^z)$ becomes an exponentially-convex-accuracy function:

$$g(e^z) = \frac{e^z}{e-1} - \frac{1}{e-1} = \frac{e^z - 1}{e-1}. \quad (11)$$

Then, upon plugging (11) into (7), entry u_{ij} of payoff matrix (4) is calculated as

$$u_{ij} = a \cdot \frac{e^{x_i} - 1}{e-1} - a \cdot \frac{e^{y_j} - 1}{e-1} + \\ +a^2 \cdot \frac{e^{x_i} - 1}{e-1} \cdot \frac{e^{y_j} - 1}{e-1} \cdot \mathrm{sign}\,(y_j - x_i) = \\ = a \cdot \frac{e^{x_i} - e^{y_j}}{e-1} + \\ +a^2 \cdot \frac{(e^{x_i} - 1)(e^{y_j} - 1)}{(e-1)^2} \cdot \mathrm{sign}\,(y_j - x_i) \\ \text{for } i = \overline{1,\ N} \text{ and } j = \overline{1,\ N}. \quad (12)$$

Hence, the general goal is to determine optimal time moments for the duelist in uniform 1-bullet silent duel (1) by (2) — (4) and (12). In this paper, the particular goal is to determine whether and when the last time moment $t_N = 1$ is optimal in such a duel. The conditions of the last time moment optimality can be set to force the duelist to act the latest possible, which is quite useful in some blockchain settings, where participants (e. g., validators or miners) choose when to attempt block proposal or transaction insertion under uncertainty [9], [10]. It is quite noteworthy that

$$u_{1j} = a \cdot \frac{1 - e^{y_j}}{e-1} + a^2 \cdot \frac{(1-1)(e^{y_j} - 1)}{(e-1)^2} = \\ = a \cdot \frac{1 - e^{y_j}}{e-1} < 0 \ \ \forall j = \overline{2,\ N}$$

and thus the starting moment $t_1 = 0$ is never optimal in such a duel, regardless of the number of time moments and the scaled payoff exponential rate (determined by accuracy factor a).

The most trivial duel

Obviously, it is the best to get started with the most trivial duel, whose size is the smallest. In such a duel, the duelist has the fewest possible number of time moments to shoot. The triviality, nevertheless, does influence the optimal behavior of duelists via accuracy factor a.

Theorem 1. *In the most trivial uniform 1-bullet silent duel* (1) *by* (2) — (4) *and exponentially-convex-accuracy payoffs* (12), *where* $N = 3$ *and*

$$\langle X_3,\ Y_3,\ \mathbf{U}_3 \rangle = \\ = \left\langle \left\{0,\ \frac{1}{2},\ 1\right\},\ \left\{0,\ \frac{1}{2},\ 1\right\},\ \mathbf{U}_3 \right\rangle, \qquad (13)$$

the duelist has the single optimal time moment $t_2 = \frac{1}{2}$ *by*

$$a > \frac{e - \sqrt{e}}{\sqrt{e} - 1}, \qquad (14)$$

the duelist has the single optimal time moment $t_3 = 1$ *by*

$$a \in \left(0;\ \frac{e - \sqrt{e}}{\sqrt{e} - 1}\right), \qquad (15)$$

and the duelist has two optimal time moments $t_2 = \frac{1}{2}$ *and* $t_3 = 1$ *by*

$$a = \frac{e - \sqrt{e}}{\sqrt{e} - 1}. \qquad (16)$$

Proof. For $N = 3$ the entries of the respective payoff matrix (4) are:

$$u_{12} = a \cdot \frac{e^0 - e^{\frac{1}{2}}}{e - 1} + a^2 \cdot \frac{\left(e^0 - 1\right)\left(e^{\frac{1}{2}} - 1\right)}{(e - 1)^2} = \\ = a \cdot \frac{1 - \sqrt{e}}{e - 1} = -u_{21} < 0, \qquad (17)$$

$$u_{13} = a \cdot \frac{e^0 - e^1}{e - 1} + a^2 \cdot \frac{\left(e^0 - 1\right)\left(e^1 - 1\right)}{(e - 1)^2} = \\ = a \cdot \frac{1 - e}{e - 1} = -a = -u_{31} < 0, \qquad (18)$$

$$u_{23} = a \cdot \frac{e^{\frac{1}{2}} - e^1}{e - 1} + a^2 \cdot \frac{\left(e^{\frac{1}{2}} - 1\right)\left(e^1 - 1\right)}{(e - 1)^2} = \\ = a \cdot \frac{\sqrt{e} - e}{e - 1} + a^2 \cdot \frac{\sqrt{e} - 1}{e - 1} = a \\ = \cdot \frac{\sqrt{e}\,(1 + a) - (e + a)}{e - 1} = -u_{32}. \qquad (19)$$

Hence, with (17) — (19) matrix (4) here is

$$\mathbf{U}_3 = [u_{ij}]_{3 \times 3} = \begin{bmatrix} 0 & a \cdot \dfrac{1 - \sqrt{e}}{e - 1} & -a \\ a \cdot \dfrac{1 - \sqrt{e}}{1 - e} & 0 & a \cdot \dfrac{\sqrt{e}\,(1 + a) - (e + a)}{e - 1} \\ a & a \cdot \dfrac{\sqrt{e}\,(1 + a) - (e + a)}{1 - e} & 0 \end{bmatrix}. \qquad (20)$$

The second time moment is single optimal if the second row of matrix (20) is positive, except for $u_{22} = 0$. Having $u_{21} > 0$ by inequality (17), it is so when

$$u_{23} = a \cdot \frac{\sqrt{e}\,(1 + a) - (e + a)}{e - 1} > 0. \qquad (21)$$

As $e > 1$, inequality (21) is equivalent to inequality

$$\sqrt{e}\,(1 + a) > e + a,$$

whence

$$\sqrt{e} - e > a - a\sqrt{e}$$

and inequality (14) emerges. The third time moment is single optimal if the third row of matrix (20) is positive, except for $u_{33} = 0$. Having $u_{31} = a > 0$, it is so when

$$u_{32} = a \cdot \frac{\sqrt{e}\,(1 + a) - (e + a)}{1 - e} > 0,$$

which is equivalent to inequality

$$\sqrt{e}\,(1 + a) < e + a,$$

whence condition (15) emerges. When $u_{23} = 0$, then

$$\sqrt{e}\,(1 + a) = e + a,$$

and condition (16) emerges, by which

$$u_{22} = u_{23} = u_{32} = u_{33} = 0$$

and thus time moments $t_2 = \frac{1}{2}$ and $t_3 = 1$ becomes optimal. □

So, it is quite clear that accuracy factor a definitely influences the optimal strategy of the duelist, although the impact is not that big. Indeed, in the smallest duel, when the duelist is allowed to shoot at only three time moments, the optimal choice is between the duel span middle (second moment) and the duel end (third moment). Setting the accuracy factor to irrational value (16) is hardly possible in practice (inasmuch as, e. g., finite precision of numerical representation is only possible in practical computations), so the two-moment optimality is unlikely.

Optimal time moments in bigger duels

In bigger duels, first consider optimality of the duel end time moment. It is as more convenient, as well as is going to lighten the proof of optimality of preceding time moments (including the penultimate one).

Theorem 2. *In uniform 1-bullet silent duel* (1) *by* (2) — (4) *and exponentially-convex-accuracy payoffs* (12)*, the duel end time moment $t_N = 1$ is single optimal if only*

$$a \in \left(0;\ \frac{e - e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}} - 1}\right) \ \ by\ \ N \in \mathbb{N}\backslash\{1,\ 2\}. \tag{22}$$

At

$$a = \frac{e - e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}} - 1} \ \ by\ \ N \in \mathbb{N}\backslash\{1,\ 2\} \tag{23}$$

the duelist has two optimal time moments: penultimate moment $t_{N-1} = \frac{N-2}{N-1}$ and end moment $t_N = 1$.

Proof. The duel end moment is single optimal if only the last row of matrix (4) is positive except for entry $u_{NN} = 0$:

$$u_{Nj} = a \cdot \frac{e^{x_N} - e^{y_j}}{e-1} - a^2 \cdot \frac{\left(e^{x_N} - 1\right)\left(e^{y_j} - 1\right)}{(e-1)^2} > 0$$
$$\forall\, y_j < x_N = 1 \ \text{ by } \ j = \overline{1,\ N-1}. \tag{24}$$

At $n \in \left\{\overline{2,\ N}\right\}$, function u_{nj} is decreasing with respect to index $j = \overline{1,\ n-1}$: indeed, $e^{x_n} > 1$, $e^{y_j} \geqslant 1$, and thus value

$$u_{nj} = a \cdot \frac{e^{x_n} - e^{y_j}}{e-1} - a^2 \cdot \frac{\left(e^{x_n} - 1\right)\left(e^{y_j} - 1\right)}{(e-1)^2} = a \cdot \frac{e^{x_n} - 1}{e-1} - a^2 \cdot \left(\frac{1}{a} + \frac{e^{x_n} - 1}{e-1}\right) \cdot \frac{e^{y_j} - 1}{e-1}$$

is a negatively-sloped line with respect to exponent e^{y_j}. Therefore, function u_{Nj} is decreasing with respect to index $j = \overline{1,\ N-1}$, and inequality (24) is equivalent to inequality

$$u_{N,N-1} = a \cdot \frac{e^{x_N} - e^{y_{N-1}}}{e-1} - a^2 \cdot \frac{\left(e^{x_N} - 1\right)\left(e^{y_{N-1}} - 1\right)}{(e-1)^2} > 0. \tag{25}$$

Inequality (25) is simplified to

$$u_{N,N-1} = \frac{e - e^{y_{N-1}}}{e-1} - a \cdot \frac{(e-1)\left(e^{y_{N-1}} - 1\right)}{(e-1)^2} = \frac{e - e^{y_{N-1}}}{e-1} - \frac{a \cdot \left(e^{y_{N-1}} - 1\right)}{e-1} = \frac{e - e^{y_{N-1}} \cdot (1+a) + a}{e-1} > 0 \tag{26}$$

whence

$$e - e^{\frac{N-2}{N-1}} \cdot (1+a) + a > 0,$$

$$a \cdot \left(e^{\frac{N-2}{N-1}} - 1\right) < e - e^{\frac{N-2}{N-1}},$$

$$a < \frac{e - e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}} - 1}. \tag{27}$$

Inequality (27) means that the end moment $t_N = 1$ is single optimal by (22).

If (23) is true, then it follows from (24) — (27) that

$$u_{N,N-1} = u_{NN} = u_{N-1,N} = u_{N-1,N-1} = 0, \tag{28}$$

and still the end moment $t_N = 1$ is optimal. The penultimate moment $t_{N-1} = \frac{N-2}{N-1}$ is optimal if $u_{N-1,N-2} \geqslant 0$ due to function $u_{N-1,j}$ is decreasing with respect to index $j = \overline{1,\ N-2}$. So,

$$u_{N-1,N-2} = a \cdot \frac{e^{x_{N-1}} - e^{y_{N-2}}}{e-1} - a^2 \cdot \frac{\left(e^{x_{N-1}} - 1\right)\left(e^{y_{N-2}} - 1\right)}{(e-1)^2} =$$
$$= \frac{e - e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}} - 1} \cdot \left(\frac{e^{\frac{N-2}{N-1}} - e^{\frac{N-3}{N-1}}}{e-1} - \frac{e - e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}} - 1} \cdot \frac{\left(e^{\frac{N-2}{N-1}} - 1\right)\left(e^{\frac{N-3}{N-1}} - 1\right)}{(e-1)^2}\right) =$$

$$=\frac{e-e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}}-1}\cdot\frac{\left(e^{\frac{N-2}{N-1}}-e^{\frac{N-3}{N-1}}\right)(e-1)-\left(e-e^{\frac{N-2}{N-1}}\right)\left(e^{\frac{N-3}{N-1}}-1\right)}{(e-1)^2}=$$

$$=\frac{e-e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}}-1}\cdot\frac{e\cdot e^{\frac{N-2}{N-1}}-e\cdot e^{\frac{N-3}{N-1}}-e^{\frac{N-2}{N-1}}+e^{\frac{N-3}{N-1}}-e\cdot e^{\frac{N-3}{N-1}}+e^{\frac{N-2}{N-1}}\cdot e^{\frac{N-3}{N-1}}+e-e^{\frac{N-2}{N-1}}}{(e-1)^2}=$$

$$=\frac{e-e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}}-1}\cdot\frac{e\cdot e^{\frac{N-2}{N-1}}-2e^{\frac{N-2}{N-1}}+e^{\frac{N-3}{N-1}}+e^{\frac{N-2}{N-1}}\cdot e^{\frac{N-3}{N-1}}+e-2e\cdot e^{\frac{N-3}{N-1}}}{(e-1)^2}=$$

$$=\frac{e-e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}}-1}\cdot\frac{e\cdot e^{\frac{N-2}{N-1}}-2e^{\frac{N-2}{N-1}}+e^{\frac{N-3}{N-1}}\cdot\left(1+e^{\frac{N-2}{N-1}}+e^{\frac{2}{N-1}}-2e\right)}{(e-1)^2}=$$

$$=e^{\frac{N-3}{N-1}}\cdot\frac{e-e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}}-1}\cdot\frac{e\cdot e^{\frac{1}{N-1}}-2e^{\frac{1}{N-1}}+1+e^{\frac{N-2}{N-1}}+e^{\frac{2}{N-1}}-2e}{(e-1)^2}. \quad (29)$$

Clearly,

$$e-e^{\frac{N-2}{N-1}}>0 \text{ and } e^{\frac{N-2}{N-1}}-1>0,$$

so the last term in (29) is nonnegative if

$$e\cdot e^{\frac{1}{N-1}}-2e^{\frac{1}{N-1}}+1+e^{\frac{N-2}{N-1}}+e^{\frac{2}{N-1}}-2e\geqslant 0. \quad (30)$$

The left side of inequality (30) can be written as

$$e\cdot e^{\frac{1}{N-1}}-2e^{\frac{1}{N-1}}+1+$$
$$+e^{\frac{N-2}{N-1}}+e^{\frac{2}{N-1}}-2e=$$
$$=\varphi_1(N)+\varphi_2(N),$$

where

$$\varphi_1(N)=e\cdot e^{\frac{1}{N-1}}+e^{\frac{N-2}{N-1}}-2e \quad (31)$$

and

$$\varphi_2(N)=1+e^{\frac{2}{N-1}}-2e^{\frac{1}{N-1}}. \quad (32)$$

The first derivative of function (31) is

$$\frac{d\varphi_1}{dN}=-\frac{e\cdot e^{\frac{1}{N-1}}}{(N-1)^2}+$$
$$+e^{\frac{N-2}{N-1}}\cdot\left(\frac{1}{N-1}-\frac{N-2}{(N-1)^2}\right)=$$
$$=-\frac{e\cdot e^{\frac{1}{N-1}}}{(N-1)^2}+e^{\frac{N-2}{N-1}}\cdot\frac{1}{(N-1)^2}=$$
$$=\frac{e^{\frac{N-2}{N-1}}-e^{\frac{N}{N-1}}}{(N-1)^2}=$$
$$=\frac{e^{\frac{N-2}{N-1}}}{(N-1)^2}\cdot\left(1-e^2\right)<0 \ \ \forall N>2,$$

so (31) is a decreasing function. Its minimal value is

$$\min_{N>2}\varphi_1(N)=\lim_{N\to\infty}\varphi_1(N)=$$
$$=\lim_{N\to\infty}\left(e\cdot e^{\frac{1}{N-1}}+e^{\frac{N-2}{N-1}}-2e\right)=$$
$$=e\cdot e^0+e^1-2e=0$$

and this minimum is not reached. So,

$$\varphi_1(N)>0 \ \ \forall N>2. \quad (33)$$

The first derivative of function (32) is

$$\frac{d\varphi_2}{dN}=-\frac{2e^{\frac{2}{N-1}}}{(N-1)^2}+\frac{2e^{\frac{1}{N-1}}}{(N-1)^2}=$$
$$=\frac{2e^{\frac{1}{N-1}}}{(N-1)^2}\cdot\left(1-e^{\frac{1}{N-1}}\right)<0 \ \ \forall N>2$$

as $e^{\frac{1}{N-1}}>1$, so (32) is a decreasing function. Its minimal value is

$$\min_{N>2}\varphi_2(N)=\lim_{N\to\infty}\varphi_2(N)=$$
$$=\lim_{N\to\infty}\left(1+e^{\frac{2}{N-1}}-2e^{\frac{1}{N-1}}\right)=$$
$$=1+e^0-2e^0=0$$

and this minimum is not reached. So,

$$\varphi_2(N)>0 \ \ \forall N>2. \quad (34)$$

Therefore, due to (33) and (34), inequality (30) holds even strictly. This means that $u_{N-1,N-2}>$ >0 and thus the last two rows of matrix (4) are positive except for entries (28), whence the penultimate moment $t_{N-1}=\frac{N-2}{N-1}$ is optimal by (23) along with the end moment $t_N=1$, and there are no other optimal time moments at the duelist. □

It is worth noting that Theorem 2 includes Theorem 1. A corollary from Theorem 2 is that if the accuracy factor scaling the reward exceeds marginal value (23), then shooting at the very end of the duel is not optimal.

Conclusion

The last time moment is optimal in uniform 1-bullet silent duel (1) by (2)—(4) and exponentially-convex-accuracy payoffs (12) only when the accuracy factor does not exceed marginal value (23). If the accuracy factor is dropped below marginal value (23), then the last time moment is single optimal. If the accuracy factor is exactly equal to marginal value (23), the duelist has two optimal time moments: the penultimate and last one. In future work, it would be worth-while to determine whether preceding time moments are optimal in such duels.

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Романюк В. В.

ОПТИМАЛЬНІСТЬ ОСТАННЬОГО МОМЕНТУ ЧАСУ В РІВНОМІРНІЙ ОДНОКУЛЬОВІЙ БЕЗШУМНІЙ ДУЕЛІ З МАСШТАБОВАНОЮ ЕКСПОНЕНЦІАЛЬНО-ОПУКЛОЮ ВЛУЧНІСТЮ

Рівномірна однокульова безшумна дуель з масштабованою експоненціально-опуклою влучністю виграшів є симетричною матричною грою, чиє оптимальне значення дорівнює 0, а кожен з дуелянтів має однакову оптимальну поведінку, хай вона у чистих або у змішаних стратегіях. Такі дуелі моделюють двосторонню змагальницьку взаємодію, де метою є здобуття винагороди за якомога кращого рішення у квантованому часі. Доведено, що останній момент часу є оптимальним у дуелі з N моментами часу лише тоді, коли коефіцієнт влучності не перевищує граничного значення $\frac{e-e^{\frac{N-2}{N-1}}}{e^{\frac{N-2}{N-1}}-1}$. Якщо коефіцієнт влучності падає нижче цього граничного значення, останній момент часу є єдиним оптимальним. Якщо коефіцієнт влучності точно рівний цьому граничному значенню, дуелянт має два оптимальні моменти часу: передостанній та останній. Умови оптимальності останнього моменту часу можуть накладатися для того, щоб змусити дуелянта діяти якомога пізніше, що є достатньо корисним у деяких налаштуваннях блокчейну, де учасники (наприклад, валідатори або майнери) обирають, коли спробувати пропонувати блок або вставку трансакції за умов невизначеності.

Ключові слова: рівномірна однокульова безшумна дуель, масштабована влучність, експоненціально-опукла влучність, матрична гра, оптимальність останнього моменту часу.

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