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ANALYSIS OF WAVE PROPAGATION CONDITIONS IN A TWO-LAYER HYDRODYNAMIC SYSTEM WITH A FREE SURFACE

The study examines the problem of the propagation of internal and surface waves in a two-layer hydrodynamic system "a half-space - a layer - a layer with a free surface". A mathematical model in a linear approximation is presented. The research problem is formulated under the assumption that the fluids are ideal and incompressible. The mathematical formulation of the problem is given in a dimensionless form. Expressions for the deviation of the contact interface $\eta_1(x,t)$ and the free surface $\eta_2(x,t)$ in the form of traveling waves are found. Expressions for the potentials $\phi_1(x,z,t)$ and $\phi_2(x,z,t)$, whose gradients describe the propagation velocities in the layers Ω_1 and Ω_2 respectively, are obtained in an analytical form. A dispersion relation that connects the wave number and the wave propagation frequency is derived. The roots of the dispersion relation, which are the frequencies of wave propagation on the contact interface and on the free surface, are found. An analysis of the roots of the dispersion relation depending on the geometric and physical parameters of the system is carried out. In particular, the dependence of the wave propagation frequencies on the wave number without considering surface tension is analyzed.

The conducted research indicates that in the absence of surface tension ($T_1 = T_2 = 0$), the density ratio ρ acts as a defining parameter that governs both the quantitative and qualitative characteristics of the wave modes in the considered system. A transition from the classical state of the system with clearly separated fast surface and slow internal modes to a regime of their inversion was identified, which is a significant result for a deeper understanding of the dynamics of strongly stratified fluids.

The consideration of surface tension forces reveals a complex interaction between the effects of density stratification and capillarity. Capillary forces lead to a substantial increase in wave frequencies and can become a dominant factor for internal modes, effectively neutralizing the influence of density changes. At the same time, it has been established that the density ratio ρ retains its role as the key parameter that determines the qualitative structure of the modes, including the possibility of their complete inversion under conditions of strong fluid stratification.

Keywords: wave propagation, two-layer system, dispersion relation.

Introduction

The investigation of wave propagation conditions in stratified hydrodynamic systems is based on the analysis of the dispersion relation. Such studies are an important component of modern research into wave motions in fluids.

In article [1] investigates the problem of wave propagation in a hydrodynamic system consisting of a layer with a rigid bottom and a layer with a free surface. The roots of the dispersion equation are analyzed for various values of the density ratio. In the limiting cases, the correspondence of the obtained roots to previously known results is shown. The existence of two linearly independent solutions for the first-order approximation problem is demonstrated, and the shapes of the free surface and the interface are also investigated.

In [2] models nonlinear internal waves in an ocean of great depth. The ocean is assumed to be composed of three homogeneous fluid layers of different densities in a stable stratified configuration. Based on the Ablowitz-Fokas-Musslimani formulation for irrotational flows, strongly nonlinear and weakly nonlinear models are developed for the "shallow-shallow-deep" and "deep-shallow-deep" scenarios. Internal solitary waves are computed using numerical iteration schemes, and their global bifurcation diagrams are obtained by a numerical continuation method and compared for different models. For the "shallow-shallow-deep" case, both mode-1 and mode-2 internal solitary waves can be found, and on the mode-2 branch, a pulse broadening phenomenon resulting in conjugate flows is observed. While in the "deep-shallow-deep" situation, only mode-2 solitary waves can be obtained. The existence and stability of mode-2 internal solitary waves are confirmed by solving the primitive equations based on the MITgcm model.

In [3] investigates the weakly nonlinear problem of internal wave packet propagation in a "half-space – layer – layer with a rigid lid" system. Based on this problem, three linear problems are constructed for the scale components of the velocity potentials and the displacements of the interfaces. The conditions for wave propagation in the first approximation are established for different density ratios in the hydrodynamic system, and the dependences of possible frequencies on the top layer thickness and the wave number are analyzed. The influence of the presence of surface tension at the interfaces is investigated.

Effects of surface tension reduction on wind-wave growth are investigated using direct numerical simulations of air-water two-phase turbulent flows [4]. The incompressible Navier-Stokes equations for the air and water sides are solved using an arbitrary Lagrangian-Eulerian method with boundary-fitted moving grids. The growth of finite-amplitude, non-breaking gravity-capillary waves, with a wavelength of less than 0.07 m, is simulated for two cases of different surface tensions under a low wind speed condition of several meters per second. The results show that the significant wave height for the smaller surface tension case increases faster than for the larger surface tension case. Analysis of energy fluxes for gravitational and capillary wave scales shows that when the surface tension is reduced, the energy transfer from the significant gravity waves to the capillary waves decreases, and the significant waves accumulate more energy supplied by the wind. This results in faster wave growth for the smaller surface tension case. To support this conjecture, the effect of surface tension is compared with laboratory experiments in a small wind-wave tank.

In [5] the evolution of a wave-like front perturbed by space-correlated disorder was studied. In addition, the generic solution for the field mean-value was presented as a series expansion in Terwiel's cumulant operators. This infinite series truncates due to the algebra of 'naked' Terwiel's cumulants when these cumulants are associated with a space exponential-correlated symmetric binary disorder. An equivalent approach is applied to study the dispersion relation for one-dimensional surface gravity waves propagating over an irregular bottom. The theory is based on the study of the mean value of plane-wave-like Fourier modes for the propagation and damping of surface waves on a random bottom.

In article [6] investigated the problem of wave propagation in a three-layer hydrodynamic system described as 'a layer with a rigid bottom–a layer–a layer with a rigid lid'. For the first approximation, the dispersion relation and its two pairs of roots are obtained. Expressions for the amplitude ratios of the interface displacements, corresponding to the roots of the dispersion equation, are derived. The dependences of these amplitude ratios on various physical parameters are graphically illustrated and analyzed.

Problem statement

The problem of the propagation of two-dimensional waves at the interface (internal waves) and on the free surface (surface waves) is investigated within a hydrodynamic system described as "half-space - layer with a free surface".

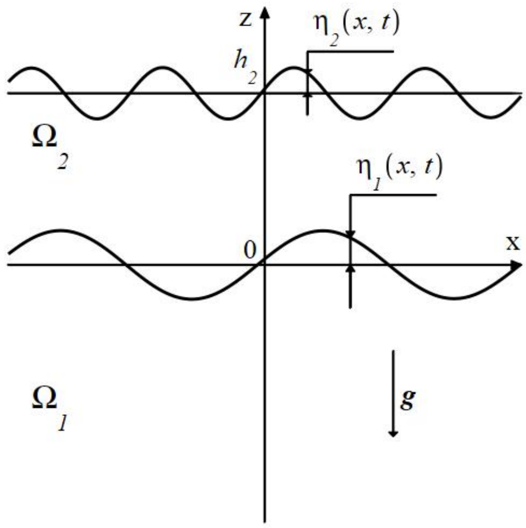

Figure 1. Problem statement.

The lower layer, $\Omega_1 = \{(x,z) : |x| < \infty, -\infty < < z < 0\}$, has a density of ρ_1, and the upper layer, $\Omega_2 = \{(x,z) : |x| < \infty, 0 < z < h_2\}$, has a density of ρ_2. The layers are separated by the interface $z = \eta_1(x,t)$, and the upper layer is bounded from above by the free surface $z = \eta_2(x,t)$. The forces of surface tension at the interface, T_1, and on the free surface, T_2, are taken into account. The force of gravity is directed perpendicular to the interface in the negative z-direction, and the fluids are considered to be incompressible (Fig.1).

Wave propagation velocities in the respective domains are expressed in terms of the gradients of the potentials ϕ_1 in Ω_1 and ϕ_2 in Ω_2. The mathematical formulation of the problem under study in the linear approximation using dimensionless variables is presented below.

$$\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial z^2} = 0 \text{ in } \Omega_i,$$

kinematic conditions

$$\frac{\partial \eta_1}{\partial t} - \frac{\partial \phi_i}{\partial z} = 0 \text{ at } z = 0,$$

$$\frac{\partial \eta_2}{\partial t} - \frac{\partial \phi_2}{\partial z} = 0 \text{ at } z = h_2,$$

dynamic conditions

$$\frac{\partial \phi_1}{\partial t} - \rho\frac{\partial \phi_2}{\partial t} + (1-\rho)\eta_1 = T_1\frac{\partial^2 \eta_1}{\partial x^2} \text{ at } z = 0,$$

$$\frac{\partial \phi_2}{\partial t} + \eta_2 = T_2\frac{\partial^2 \eta_2}{\partial x^2} \text{ at } z = h_2,$$

boundary condition at infinity

$$|\nabla \phi_1| \to 0 \text{ at } z \to -\infty,$$

where $\rho = \rho_1/\rho_2$ is the density ratio.

In the following section, the solutions of the linear approximation will be presented, and the derived dispersion relation will be provided.

Solutions and the dispersion relation

For the mathematical formulation presented above, solutions have been obtained in the form of traveling waves. The displacements of the interface and the free surface are found in the following form:

$$\eta_1 = A\exp(i\theta + kz) + c.c.,$$

$$\eta_2 = (\omega^2\rho)^{-1}[(T_1k^3 - k\rho + k - \omega^2)\sinh(kh_2) -$$
$$-\omega^2\rho\cosh(kh_2)]A\exp(i\theta + kz) + c.c.,$$

where A is the wave amplitude, k is the wave number, ω is the wave frequency, $\theta = kx - \omega t$, and *c.c.* denotes the complex conjugate of the preceding term.

The expressions for the potentials are as follows:

$$\phi_1 = -\frac{i\omega}{k}A\exp(i\theta + kz) + c.c.,$$

$$\phi_2 = \frac{i}{k\omega\rho}[(T_1k^3 - k\rho + k - \omega^2)\cosh(kz) -$$
$$-\omega^2\rho\sinh(kz)]A\exp(i\theta) + c.c.$$

Based on the solutions of the linear approximation, a dispersion relation that relates the wave frequency and the wave number has been derived. The dispersion relation is obtained in the following form:

$$\omega^2 = (k + T_2k^3)\frac{c_1 - c_2\coth(kh_2)}{c_2 - c_1\coth(kh_2)},$$

where $c_1 = T_1k^3 - k\rho + k - \omega^2$, $c_2 = \omega^2\rho$.

The dispersion relation can be expressed in the form of a biquadratic equation for the wave frequency ω as follows:

$$a\omega^4 + b\omega^2 + c = 0,$$

where

$$a = \rho + \coth(kh_2),$$
$$b = (1 + \rho\coth(kh_2))(T_2k^3 + k) -$$
$$-\coth(kh_2)(T_1k^3 - k\rho + k),$$
$$c = -(T_2k^3 + k)(T_1k^3 - k\rho + k).$$

The equation has two pairs of roots:

$$\omega_{1,3} = \mp\sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}},$$

$$\omega_{2,4} = \mp\sqrt{\frac{-b + \sqrt{b^2 - 4ac}}{2a}},$$

which represent the frequencies of waves that can propagate in the system under study. In the following section, an analysis of these roots will be performed as a function of the geometric and physical parameters of the system.

Analysis of the roots of the dispersion relation

An analysis of the dispersion relations for two-dimensional waves in the hydrodynamic system 'a heavy fluid half-space – a layer of a lighter fluid with a free surface' is presented. The influence of the density ratio $\rho = \rho_2/\rho_1$ on the frequency characteristics of the waves $\omega(k)$ was studied in the absence of surface tension forces ($T_1 = T_2 = 0$) and for a fixed dimensionless thickness of the upper layer $h_2 = 1$. The dispersion relation for this system has two branches of real solutions $\omega(k)$, corresponding to two wave modes: $\omega_1(k)$ - blue color and $\omega_2(k)$ - red color.

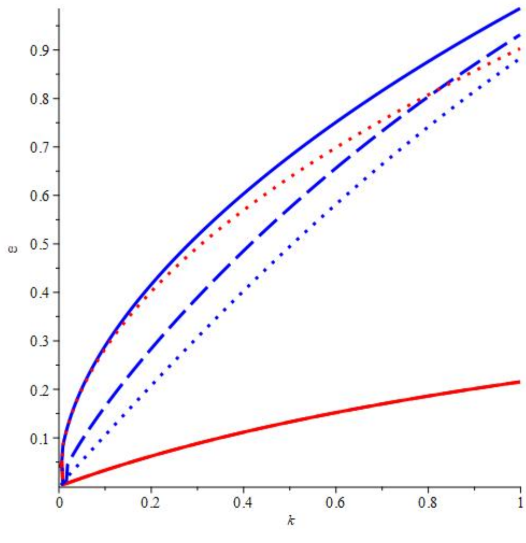

Figure 2. Dispersion curves without surface tension: $\rho = 0.9$ - solid lines, $\rho = 0.5$ - dash lines, $\rho = 0.1$ - dot lines.

In the case where the fluid densities are close ($\rho = 0.9$), the system exhibits classical behavior (Fig.2). There are two clearly distinct modes: a high-frequency mode, which corresponds to surface (barotropic) waves, and a low-frequency mode, which corresponds to internal (baroclinic) waves at the contact interface. The restoring force for the internal waves is the reduced gravity, proportional to the density difference $(1 - \rho)$, which explains their significantly lower frequencies compared to the surface waves.

With a decrease of the parameter ρ to 0.5, corresponding to an increase in the density contrast, the frequency of the surface mode undergoes only minor changes, confirming its weak sensitivity to internal stratification. In contrast, the behavior of the internal mode shows a non-trivial result. Its values remain at practically the same level as in the previous case. This indicates the presence of complex inertial effects in the system's dynamics, which compensate for the increase in the restoring force for the given parameters.

A further decrease of the density ratio to $\rho = 0.1$ leads to a qualitative restructuring of the wave dynamics. The effect of mode swapping is observed: the branch that previously corresponded to the slow internal mode now becomes high-frequency, and vice versa. In this regime, when the upper layer becomes extremely light, the contact interface between the fluids begins to behave like a free surface for the heavy lower half-space, which causes the high frequencies of the corresponding mode. At the same time, the waves on the free surface of the light layer itself become dynamically slower.

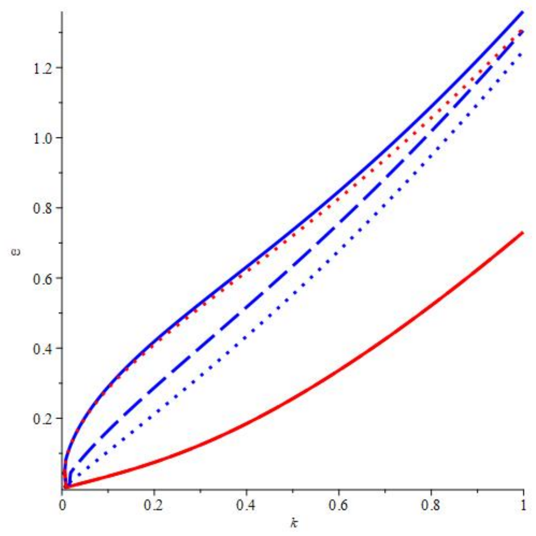

Figure 3. Dispersion curves with surface tension: $\rho = 0.9$ - solid lines, $\rho = 0.5$ - dash lines, $\rho = 0.1$ - dot lines.

In the case of close densities ($\rho = 0.9$), the presence of surface tension leads to a significant increase in the frequencies of both modes compared to the purely gravitational case. This transforms the waves into gravity-capillary waves, where capillary forces, proportional to k^3, become a substantial restoring force, especially for short waves. This effect is most pronounced for the internal mode ω_2, whose frequency increases severalfold, which indicates the significant influence of the tension at the interface (Fig.3).

When the parameter ρ is decreased to 0.5, corresponding to an increase in the density contrast, the dispersion curves are nearly identical to the previous case. Despite a five-fold increase in the gravitational restoring force for the internal mode, its frequency ω_2 remains practically unchanged. This indicates that for the given parameters, the capillary force at the interface becomes the dominant factor in shaping the dynamics of the internal waves. The contribution from the gravitational component, related to the density difference, becomes secondary; therefore, the change in ρ has almost no effect on the final frequency.

A further decrease of the density ratio to $\rho = 0.1$ leads to a qualitative restructuring of the wave dynamics, analogous to that observed in the case without surface tension. Mode inversion occurs, and the ω_2 branch becomes high-frequency, exceeding the ω_1 branch. This fundamental phenomenon, caused by strong stratification, persists even in the presence of capillary effects. Surface tension in this regime acts as an additional amplifying factor, further increasing the frequencies of both already-restructured modes. Thus, the waves at the interface (ω_2) become the fastest in the system, as their dynamics are determined by the combined action of the full force of gravity and significant surface tension.

Conclusions

The analysis shows that, in the absence of surface tension on both surfaces, the density ratio ρ is a key parameter that controls not only the quantitative but also the qualitative characteristics of the wave modes in the system. A transition from the classical configuration with clearly defined fast surface and slow internal modes to a regime in which their inversion occurs is demonstrated, which is an important result for understanding the dynamics of strongly stratified fluids.

In the presence of surface tension forces, the analysis reveals a complex interaction between density stratification and capillary effects. Surface tension significantly increases the wave frequencies and can become the dominant factor for internal modes, nullifying the effect of density changes. At the same time, the density ratio ρ remains the key parameter that determines the qualitative structure of the modes, up to their complete inversion under strong fluid stratification.

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АНАЛІЗ УМОВ ПОШИРЕННЯ ХВИЛЬ У ДВОШАРОВІЙ ГІДРОДИНАМІЧНІЙ СИСТЕМІ З ВІЛЬНОЮ ПОВЕРХНЕЮ

У дослідженні розглянуто задачу про поширення внутрішніх та поверхневих хвиль в двошаровій гідродинамічній системі «півпростір - шар - шар з вільною поверхнею». Представлено математичну модель в лінійному наближенні. Проблема дослідження сформульована в припущенні, що рідини є ідеальними та нестисливими. Математична постановка задачі наведена в безрозмірному вигляді. Знайдено вирази для відхилення поверхні контакту $\eta_1(x,t)$ та вільної поверхні $\eta_2(x,t)$ у вигляді біжучих хвиль. Отримано в аналітичному вигляді вирази для потенціалів $\phi_1(x,z,t)$ та $\phi_2(x,z,t)$, градієнти яких описують швидкості поширення в шарах Ω_1 та Ω_2 відповідно. Виведено дисперсійне співвідношення, яке пов'язує між собою хвильове число та частоту поширення хвилі. Знайдено корені дисперсійного співвідношення, які є частотами поширення хвиль на поверхні контакту та на вільній поверхні. Проведено аналіз коренів дисперсійного співвідношення в залежності від геометричних та фізичних параметрів системи. Зокрема, проаналізовано залежність частот поширення хвиль від хвильового числа без урахування поверхневого натягу.

Проведене дослідження свідчить, що в умовах відсутності поверхневого натягу ($T_1 = T_2 = 0$) відношення густин ρ виступає як визначальний параметр, що керує як кількісними, так і якісними характеристиками хвильових мод у розглядуваній системі. Було виявлено перехід від класичного стану системи з чітко розділеними швидкою поверхневою та повільною внутрішньою модами до режиму їхньої інверсії, що є суттєвим результатом для глибшого розуміння динаміки рідин із значною стратифікацією.

Врахування сил поверхневого натягу розкриває комплексну взаємодію між ефектами стратифікації за густиною та капілярністю. Капілярні сили призводять до істотного зростання хвильових частот і можуть стати домінантним фактором для внутрішніх мод, фактично нейтралізуючи вплив змін густини. Разом з тим встановлено, що відношення густин ρ зберігає свою роль ключового параметра, який визначає якісну структуру мод, включно з можливістю їхньої повної інверсії в умовах сильної стратифікації рідини.

Ключові слова: поширення хвиль, двошарова система, дисперсійне співвідношення.

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